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EFFECTIVE POINT SEPARATION ON PLURI-CANONICAL SYSTEMS ON 3-FOLDS

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This is a report on our joint work with Jungkai Alfred Chen and Meng Chen [1].

One main goal of algebraic geometry is to classify algebraic varieties. The successful 3-dimensional MMP (see [14, 17] for example) has been attracting more and more mathematicians to the study of algebraic 3-folds. In this paper, we restrict our interest to projective minimal Gorenstein 3-folds X of general type where there still remain many open problems.

Denote by K_X the canonical divisor and $\Phi_m := \Phi_{|mK_X|}$ the m-canonical map. There has been a lot of work along the line of the canonical classification. For instance, when X is a smooth 3-fold of general type with pluri-genus $h^0(X, kK_X) \geq 2$, in [15], as an application to his research on higher direct images of dualizing sheaves, Kollár proved that Φ_m, with $m = 11k + 5$, is birational onto its image. This result was improved by the second author [5] to include the cases m with $m \geq 5k + 6$; see also [7], [9] for results when some additional restrictions (like bigger $p_g(X)$) are imposed.

On the other hand, for 3-folds X of general type with $q(X) > 0$, Kollár [15] first proved that Φ_{225} is birational. Recently, the first author and Hacon [4] proved that Φ_m is birational for $m \geq 7$ by using the Fourier-Mukai transform. Moreover, Luo [20], [21] has some results for 3-folds of general type with $h^2(\mathcal{O}_X) > 0$.

Now for minimal and smooth projective 3-folds, it has been established that Φ_m ($m \geq 6$) is a birational morphism onto its image after 20 years of research, by Wilson [26] in 1980, Benveniste [2] in 1986 ($m \geq 8$), Matsuki [22] in 1986 ($m = 7$), the second author [6] in 1998 ($m = 6$) and independently by Lee [18], [19] in 1999-2000 ($m = 6$; and also the base point freeness of m-canonical system for $m \geq 4$).

The aim of this paper is to prove the following:

Theorem 0.1. *Let X be a projective minimal Gorenstein 3-fold of general type with canonical singularities. Then the m-canonical map Φ_m is a birational morphism onto its image for all $m \geq 5$.*

This result is unexpected previously. The difficulty lies in the case with smaller $p_g(X)$ or K_X^3. One reason to account for this is that the non-birationality of the 4-canonical system for surfaces may happen when they have smaller p_g or K^2 (see Bombieri [3]), whence a naive induction on the dimension does not work.

Nevertheless, there is also evidence supporting the birationality of Φ_5 for Gorenstein minimal 3-folds X of general type. For instance, one sees that $K_X^3 \geq 2$ for minimal and smooth X. So an analogy of Fujita's conjecture would predict that $|5K_X|$ gives a birational map. We recall that Fujita's conjecture (the freeness part) has been proved by Fujita, Ein-Lazarsfeld [10] and Kawamta [12] when $\dim X \leq 4$.

Remark 0.2. The numerical bound "5" in Theorem 0.1 is optimal. There are plenty of supporting examples. For instance, let $f : V \longrightarrow B$ be any fibration where V is a smooth projective 3-fold of general type and B a smooth curve. Assume that a general fiber of f has a minimal model S with $K_S^2 = 1$ and $p_g(S) = 2$. (For example, take the product.) Then $\Phi_{|4K_V|}$ is evidently not birational (see [3]).

Remark 0.3. **Reduction to birationality.** According to [6] or [18], to prove Theorem 0.1, we only need to verify the statement in the case $m = 5$. On the other hand, the results in [18, 19] show that $|mK_X|$ is base point free for $m \geq 4$. So it is sufficient for us to verify the birationality of $|5K_X|$ in this paper.

Remark 0.4. **Reduction to factorial models.** According to the work of M. Reid [23] and Y. Kawamata [13] (Lemma 5.1), there is a minimal model Y with a birational morphism $\nu : Y \longrightarrow X$ such that $K_Y = \nu^*(K_X)$ and that Y is factorial with at worst terminal singularities. Thus it is sufficient for us to prove Theorem 0.1 for minimal factorial models.

We now sketch the proof of the main theorem. By the Riemann-Roch theorem of Reid, one sees that the 2nd purigenus $P_2(X) \geq 4$. Let $\Phi_m := \Phi_{|mK_X|}$. When $m = 2$, we set $d = \dim \Phi_2(X)$. Then $d = 1, 2, 3$. The case $p_g(X) \geq 2$ or $d = 1$ is easy and has been partly treated in Chen [9], Theorem 3.3, and Chen [7]. We consider the case $d = 3$ (the case $d = 2$ is a bit different). Set $|2K_X| = |M| + Z$ with Z the fixed part. Let $\pi : X' \to X$ be the composition of (minimal) resolutions of Sing(X) and base points of $|M|$ so that the movable part $|S|$ of $|2K_{X'}|$ is base point free. Set $\pi^*M = S + \Delta$. Note that Sing(X) is a finite set (see [17], Corollary 5.18). We may write $E_\pi = \Delta' + \Delta''$ where Δ' (resp. Δ'') lies (resp. does not lie) over the base locus of $|M|$. One has $\mathrm{Supp}(\Delta) = \mathrm{Supp}(\Delta')$.

A general member S is a smooth surface of general type. We shall reduce to the surface case and apply Reider's result. Let $L = \pi^*(K_X)|S$, which is a nef and big Cartier divisor on S. The Kawamata-Viehweg vanishing theorem implies:

$$|K_{X'} + 2\pi^*(K_X) + S|_{|S} = |K_S + 2L|.$$

Clearly, $|5K_X|$ separates points on different S. So it is sufficient to show that the complete linear system $|K_S + 2L|$ on S gives rise to a birational map. Assume that $L^2 \geq 3$ (the case $L^2 \leq 2$ needs a bit more care). Then $(2L)^2 \geq 12$ and we can apply Reider's result. Suppose the contrary that $|K_S + 2L|$ does not define a birational map. Then there is a free pencil $|C|$ on S such that $L.C = 1$. Let $R = S_{|S}$. Then $R \leq 2L$. Clearly, $|R|$ is base point free and is not composed of a pencil. Thus $\dim(\Phi_{|R|}(C)) = 1$. Since C lies in an algebraic family and S is of general type, we have $g(C) \geq 2$. Noting that $h^0(C, R_{|C}) \geq 2$, the Riemann-Roch theorem on C and Clifford's theorem on C easily imply $R \cdot C \geq 2$. Since $R \cdot C \leq 2L \cdot C = 2$, one must have $R \cdot C = 2$. Noting that

$$2L = S_{|S} + \triangle_{|S} + \pi^*(Z)_{|S}$$

and C is nef, we have $\triangle_{|S} \cdot C = 0$. This implies that $\triangle'_{|S} \cdot C = 0$. Note also that $\triangle''_{|S} = 0$ for general S. We get $(E_\pi)_{|S} \cdot C = 0$. Therefore

$$K_S \cdot C = (K_{X'} + S)_{|S} \cdot C = \pi^*(K_X)_{|S} \cdot C + (E_\pi)_{|S} \cdot C + S_{|S} \cdot C = 3,$$

an odd integer. This is impossible because C is a free pencil on S. Therefore, Φ_5 must be birational.

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