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Varieties with non-linear Gauss fibers

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ABSTRACT. The main purpose of this talk is an announcement of the speaker's recent result: Existence (or construction) of a projective variety whose general fiber of the Gauss map is the given projective variety. The speaker also talked about history of studies of Gauss fiber structures, and explained the linearity of Gauss fibers in characteristic 0 and differences between 0 and positive characteristics.

1. INTRODUCTION

In this paper, the base field K is an algebraically closed field and varieties are integral algebraic schemes over K.

Precisely, "Gauss fibers" mean general fibers of the Gauss maps. Definition of the Gauss map is as follows:

Definition 1.1. *Let $X \subset \mathbf{P}^N$ be a projective variety. The* Gauss map *γ on X is the rational map from X to the Grassmannian $\mathbf{G}(\dim X, N)$ such that $\gamma(p) = \mathbf{T}_p X$ for any smooth point $p \in X$, where $\mathbf{T}_p X$ is the projective embedded tangent space.*

Example 1.2. *If $X \subset \mathbf{P}^N$ is a hypersurface given by F, then*

$$\gamma = \left(\frac{\partial F}{\partial X_0} : \cdots : \frac{\partial F}{\partial X_N}\right) : X \dashrightarrow \mathbf{G}(N-1, N) \cong \mathbf{P}^{N*}.$$

We study the following natural question:

Question. What is the structure of the general fibers of γ?

(1) If $\mathrm{char} K = 0$ then general fibers are linear spaces ([1],[5],[16]). (They are one-points when $\dim = 0$.)

E. Bertini (1907) and C. Segre (1910) found many results about projective duality. The speaker does not know whether they mentioned Gauss maps, but thinks that they got essentially the linearity of fibers. About their history, refer the Kleiman's paper [11]. On the other hand, Griffiths-Harris proved this fact ([5]).

This fact, when X is a curve, implies that multiple tangent lines (which have two or more distinct tangential points at X) are only finitely many. A cone surface (which is the join of a curve and one point) and a tangent surface (which is covered by tangent lines of some curve) are concrete examples with $\dim X = 2 > 1 = \dim \gamma(X)$.

In differential geometry, a result analogous to the answer (1) is known. Let $f : U \to \mathbf{R}^3$ be a surface (U is an open subset of $\mathbf{R}^2$). The (classical) Gauss map γ is the map which assigns to a point the unit normal vector. If the rank of the differential of γ is 1 everywhere then this surface is called developable surface, which is a kind of a ruled surface. Then the Gauss map is constant on each lines. Developable surfaces can be developable onto a plane, conversely, can be made out of paper ([6]). (The speaker showed a model of a tangent surface when he talked.)

(2) If $\mathrm{char} K > 0$ then fibers may be two or more distinct points.

Example 1.3. *Let* $\mathrm{char} K = p > 2$. $YZ^{2p-1} - X^{2p} = 0 \subset \mathbf{P}^2$.

A. H. Wallace gave examples of this kind ([15]). Kleiman-Laksov also found interesting examples ([12]). It seems to be difficult to construct smooth varieties of this kind, but H. Kaji ([8],[9]), J. Rathmann ([14]) and A. Noma ([13]) constructed such varieties.

Remark 1.4. *(Zak's theorem* [16]*) Let* $X \subset \mathbf{P}^N$ *be a smooth projective variety not linear. Then the Gauss map is a finite map onto its image.*

(3) If $\mathrm{char} K = 3$ then there exists a surface whose Gauss fibers are plane smooth conics ([2]).

Example 1.5. $XZ^6 - (Y^6 + W^6)W = 0 \subset \mathbf{P}^3$.

In the speaker's best knowledge, this is the first example whose general Gauss fibers are not finite unions of linear spaces. (The speaker mistook the defining polynomial for $XZ^6 - (Y^6 + Z^6)W$ when he talked. This surface has two lines as Gauss fibers. Dr. H. Nasu pointed out this mistake at Kochi meeting next week of Kinosaki symposium. The speaker thanks him.)

More strongly, the speaker got the following answer:

(4) When charK > 0, **for any projective variety Y, there exists a projective variety X whose general fibers of the Gauss map are** Y([3]).

2. Projective duality and linearity of Gauss fibers

In this section, we introduce the general theory of projective duality. The speaker explains that the theory induces the linearity of Gauss fibers naturally when the characteristic is 0. The speaker believes that the audience or the readers can understand where he identified the differences lie between 0 and positive characteristics.

Let $X \subset \mathbf{P}^N$ be a projective variety. Let

$$CX := \overline{\{(x, H) \in X_{\mathrm{sm}} \times \mathbf{P}^{N*} | \mathbf{T}_x X \subset H\}} \subset \mathbf{P}^N \times \mathbf{P}^{N*}$$

be the conormal variety, let $p_2 : CX \to \mathbf{P}^{N*}$ be the natural projection, and let $X^* = p_2(CX)$ be the dual variety. Then we can also define CX^*, which can be considered as a subvariety of $\mathbf{P}^N \times \mathbf{P}^{N*}$.

Definition 2.1. *If $CX = CX^*$ then X is called* reflexive.

Reflexivity is stronger than duality.

Remark 2.2. *If X is reflexive, then $X^{**} = X$.*

The following theorem implies that any projective variety in characteristic 0 is reflexive.

Theorem 2.3. *(Monge-Segre-Wallace criterion [11]) Reflexivity of X is equivalent to the separability of the projection $p_2 : CX \to X^*$.*

Reflexivity implies the linearity of contact loci.

Remark 2.4. *If X is reflexive then for a general tangent hyperplane $H \in X^*$, its contact locus $X_H = \overline{\{x \in X_{\mathrm{sm}} | \mathbf{T}_x X \subset H\}} \subset X$ is a linear space.*

The proof is easy because $X_H = p_1 p_2^{-1}(H)$. The linearity of contact loci implies the linearity of Gauss fibers.

Remark 2.5. *If X is reflexive then the general fiber of the Gauss map is linear.*

The proof is easy because $\gamma^{-1}(T) = \bigcap_{T \subset H} X_H$.

We find that the linearity of Gauss fibers in characteristic 0 is induced from reflexivity, and varieties with peculiar Gauss fibers constructed by the speaker are not reflexive.

3. Main result

In this section, the speaker states the main result and gives the proof for plane curves' case. This proof is essential, and the general case is proven the same.

Theorem 3.1. *Let* $\mathrm{char} K > 0$. *For any projective variety $Y \subset \mathbf{P}^k$, there exists a projective variety $X \subset \mathbf{P}^N$ of dimension k such that for a general point $p \in X_{\mathrm{sm}}$, $Y = \gamma^{-1}(\gamma(p)) \subset \mathbf{T}_p X \cong \mathbf{P}^k$ (set-theoretically).*

Proof for plane curves $Y \subset \mathbf{P}^2$. Let $p > 0$ be the characteristic, and let $\rho_0, \rho_1, \rho_2 : \mathbf{A}^1 \to \mathbf{P}^3$ be

$$\rho_0 = (1 : 0 : u : u^p), \rho_1 = (0 : 1 : 0 : 0), \rho_2 = (0 : 0 : 1 : 0).$$

Let $\eta : \mathbf{A}^1 \times \mathbf{P}^2 \to \mathbf{P}^3$ be

$$(u) \times (1 : y_1 : y_2) \mapsto [\rho_0 + y_1\rho_1 + y_2\rho_2] = (1 : y_1 : u + y_2 : u^p).$$

We may assume that $y_1 - a$ is a local parameter at a smooth point $(1 : a : b) \in Y$. (We can always take this coordinates by linear transforms of $\mathbf{P}^2$.) Let X be the closure of $\eta(\mathbf{A}^1 \times Y)$, and let $\tau := \eta|_{\mathbf{A}^1 \times Y} : \mathbf{A}^1 \times Y \to X$. Then the following proposition holds, and completes our proof. □

Proposition 3.2. *The morphism τ is birational, and $\mathbf{T}_{\tau(u,y)}X = \eta(u \times \mathbf{P}^2)$ for a general point (u, y).*

Proof. The differentials of τ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & dy_2/dy_1 & 0 \end{pmatrix}$$

(upper row is a list of the differentials by u, lower is the differentials by y_1). We find the separability of τ by this matrix and, because τ is generically one-to-one, birationality of τ.

On the other hand, the intersection of a projective tangent space with $\mathbf{A}^3$ (whose first element is not 0) is the plane $x_3 = u^p$. This coincides with the intersection of $\eta(u \times \mathbf{P}^2)$ with $\mathbf{A}^3$. □

Explanation. The plane $\mathbf{P}^2_u \subset \mathbf{P}^3$ spanned by ρ_0, ρ_1, ρ_2 moves in $\mathbf{P}^3$ by u. $Y \subset \mathbf{P}^2$ moves in conformity to the plane $\mathbf{P}^2_u$, and constructs X. Then, for almost all point $x \in X$ the tangent space $\mathbf{T}_x X$ coincides with the plane $\mathbf{P}^2_u$ containing x. We can check easily the inseparability of η, hence the moving of $\mathbf{P}^2$ is special to positive characteristic.

Because the construction of X in the proof is parameteric, calculation of the defining polynomial of X is comparatively easy.

Example 3.3. *In the setting of the proof, let $p = 2$ and let $Y \subset \mathbf{P}^2$ be the plane curve given by $Y_0^3 + Y_1^3 + Y_2^3 = 0$. Then the defining polynomial X is*

$$X^6 + Y^6 + Z^6 + XZ^4W + X^2Z^2W^2 + X^3W^3.$$

Remark 3.4. *The proof for the general case is given in* [3]. *The base space $\mathbf{A}^1$ which moves projective planes and the how of the moving $\{\rho_i\}$ are more free and formulated to some extent.*

Furthermore, we can also construct varieties whose general fibers are two or more Y*s for the suitable moving* $\{\rho_i\}$.

4. Supplement

The idea for the construction in the proof of main theorem is based on "circular surfaces" ([6],[7]). Circular surfaces are given by moving circles continuously, and have been studied in differential geometry. Recently, they are studied from the modern viewpoint of real singularity theory ([7]). Circles are Ys in our setting. The speaker also found another new Gauss fiber structures by this idea ([4]).

Gauss maps on curves or smooth varieties are detailed in [10]. By the Zak's theorem (Remark 1.4), varieties with positive dimensional Gauss fibers are singular. Prof. H. Kaji told the speaker the following problem about one year ago: Construct varieties whose general Gauss fibers are not finite unions of linear spaces and singularities are as small as possible. However, a satisfactory answer has not been given. For example, the existence of normal surfaces whose Gauss fibers are plane smooth conics is unknown. The importance of this problem will increase.

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