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Tropical hypersurfaces and degeneration of projective toric varieties

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Abstract

We show a one-to-one correspondence between tropical hypersurfaces in tropical affine spaces and a certain degenerations of projective toric varieties with some additional structures, called log structures.

Introduction

Tropical geometry is a sort of geometry to investigate the image of algebraic varieties under rank-one valuations. So it is natural to use the min-plus algebra or tropical semiring $\mathbb{R}$ with tropical addition $\oplus$ and tropical multiplication $\odot$:

$$a \oplus b := \min(a, b), \quad a \odot b := a + b.$$

This geometry is applied to several situations, e.g., counting of algebraic curves in the projective plane [7], limits of complex curves, a study of real algebraic curves and amoeba [8].

We now apply tropical geometry to study degeneration of algebraic varieties. The purpose of this paper is to show a one-to-one correspondence between tropical hypersurfaces in tropical affine spaces, which are associated to hypersurfaces in algebraic tori, and log stable toric varieties of toric type, which are a certain degeneration of projective toric varieties (Theorems 2.3 and 2.5).

The notion of log stable toric varieties is inspired by the Mumford construction of degenerating abelian varieties [2], [9], and stable toric varieties introduced by Alexeev [1].

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1 Review of tropical geometry

In this section, we briefly recall tropical geometry.

Let K be an algebraically closed field with an additive valuation $v_K\colon K^\times \to \mathbb{R}$. We assume that $v_K(K^\times)$ is dense in $\mathbb{R}$.

Let M be a free $\mathbb{Z}$-module of rank r, and $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ the set of homomorphisms from M to $\mathbb{Z}$. We set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R})$. Let us denote by $T \cong (K^\times)^r$ an algebraic torus $\mathrm{Hom}(M, K^\times)$ with the character group M. We write $m \in M$ as x^m when m is regarded as a character of T. Note that a point $p\colon M \to K^\times$ of T defined by $x^m \mapsto x^m(p)$ gives the point $v_p := v_K \circ p\colon M \to \mathbb{R}$ of $N_\mathbb{R}$ by composing p with v_K. The map defined by $p \mapsto v_p$ is denoted by $v_T\colon T \to N_\mathbb{R}$.

1.1 Example We give some examples for algebraically closed valuation fields (K, v_K).

(1) The field of formal Puiseux series over an algebraically closed field k of characteristic zero is an algebraically closure K of the field of formal Laurent series $k((t))$. We can extend the additive valuation v with $v(t) = 1$ to K uniquely. It is known that K is the union of $k((t^{1/n}))$'s for $n \geq 1$. This algebraically closed field K is denoted by $k((t^\mathbb{Q}))$.

(2) Let $k[t^\mathbb{R}] = k[\{t^a; a \in \mathbb{R}\}]$ be the group ring over a field k associated to the additive group $\mathbb{R}$ of real numbers. We can show that the map $v_0\colon k[t^\mathbb{R}] \smallsetminus \{0\} \to \mathbb{R}$; $\sum_a c_a t^a \mapsto \min\{a; c_a \neq 0\}$ can be extended to a non-Archimedean additive valuation $v\colon k(t^\mathbb{R})^\times \to \mathbb{R}$ of the fractional field $k(t^\mathbb{R})$ of $k[t^{\mathbb{R}_{\geq 0}}]$. Let us take the algebraic closure K of $k(t^\mathbb{R})$, and an extension $\widetilde{v}\colon K^\times \to \mathbb{R}^\times$ of the valuation v. It is well-known that such an extension exists. For example, see [3, Chapitre VI] for the detail. This is an example of an algebraically closed valuation field K with $v(K^\times) = \mathbb{R}$.

(3) An algebraic closure K of a local field K_0, such as $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ has a unique extension of the valuation of K_0.

(4) In the case $K = \mathbb{C}$, we can take an Archimedean additive valuation v_K such as $v_K(z) = -\log|z|$ for $z \in \mathbb{C}^\times$.

1.2 Definition For a closed subvariety V of T, we define the *tropical variety* V^{trop} associated to V as the closure of $v_T(V) \subset N_\mathbb{R}$ with respect to the Euclidean topology of $N_\mathbb{R} = \mathbb{R}^r$.

In general, computing the points of V is difficult, but computing those of V^{trop} is rather easier by the following alternative definition. We show the equivalence of these definitions in Proposition 1.4.

1.3 Definition For an ideal I of the Laurent polynomial ring $K[M]$, the *tropical variety* $\mathbf{V}^{\mathrm{trop}}(I)$ defined by I is

$$\begin{aligned}\mathbf{V}^{\mathrm{trop}}(I) := \{v \in N_\mathbb{R};\ &\text{for every} \sum_{m \in M} a_m x^m \in I \smallsetminus \{0\}, \text{ there exist two} \\ &\text{distinct elements } m_1, m_2 \in M \text{ such that } v_K(a_{m_1}) + v(m_1) = v_K(a_{m_2}) + v(m_2) \\ &\qquad\qquad \leq v_K(a_m) + v(m) \text{ for each } m \in M\}.\end{aligned}$$

We show the following proposition, which implies the equivalence of the above definitions of tropical varieties.

1.4 Proposition *Let $v_K\colon K^\times \to \mathbb{R}$ be non-Archimedean, and suppose that there exists $t \in K^\times$ with $v_K(t) = 1$. If a closed subvariety V of T is defined by an ideal I of $K[M]$, then we have $V^{\mathrm{trop}} = \mathbf{V}^{\mathrm{trop}}(I)$ [12, Theorem 2.1].*

2 Tropical hypersurfaces and log stable toric varieties

We show a one-to-one correspondence between the tropical hypersurfaces associated to ones in an algebraic tori and log stable toric varieties.

We first recall log stable toric varieties, and polarized ones. Throughout the rest of this article, let S be a standard log point $(\operatorname{Spec} k, k^\times \oplus \mathbb{N})$, and T an algebraic torus of dimension r.

2.1 Definition (1) A *log stable toric variety* over S of dimension r is an r-dimensional log smooth variety over S whose underlying scheme is a stable toric variety in the sense of Alexeev [1], and satisfies that the support of $\mathcal{M}_X^{\mathrm{gp}}/p^*\mathcal{M}_S^{\mathrm{gp}}$ coincides with the union of $(r-1)$-dimensional T-orbits. Here p denotes the structure morphism $X \to S$. We also call such a log variety a *log STV* for short.

(2) A log STV $(X, \mathcal{M}_X)$ is *of toric type* if $H^0(X, \mathcal{M}_X^{\mathrm{gp}})/H^0(X, \mathcal{O}_X^\times)$ is free of rank $\dim X + 1$.

Throughout the rest of this article, we consider only log STVs of toric type. So log STVs mean log STVs of toric type from now on.

We next define marked polarized STVs.

2.2 Definition Let X be a log STV over S, and $p\colon X \to S$ the structure morphism of X.

(1) A *polarization* on X is a global section l of $\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^\times$ that goes to a linear equivalence class of an ample line bundle on X in $H^1(X, \mathcal{O}_X^\times)$. A log polarized stable toric variety, log PSTV for short, is a log STV with its polarization.

(2) Let M be a free $\mathbb{Z}$-module of rank r, and $\overline{M}$ the image of $H^0(X, \mathcal{M}_X^{\mathrm{gp}})$ in $H^0(X, \mathcal{M}_X^{\mathrm{gp}}/p^*\mathcal{M}_S^{\mathrm{gp}})$ by the projection. A *log STV over S with characters M* is a pair of a log PSTV (X, l) over S and an isomorphism $\imath\colon M \to \overline{M}$.

(3) Let M be a free $\mathbb{Z}$-module of rank r, and Δ an integral polytope in $M_{\mathbb{R}}$. A *log Δ-PSTV over S* is a pair of a log PSTV (X, l) over S with character $\imath\colon M \to \overline{M}$ satisfying that, $(\imath \otimes \mathbb{R})(\Delta)$ is the convex hull of

$$\{\chi^m \in \overline{M}; \chi^m l = \chi^{\tilde{m}} \text{ for a lift } \chi^{\tilde{m}} \in H^0(X, M_X/\mathcal{O}_X^\times) \text{ of } \chi^m\}$$

in $\overline{M}_{\mathbb{R}}$.

(4) A *marking* of an integral polytope Δ is a subset Q of $\Delta \cap M$ whose convex hull coincides Δ. A *marking* of a log Δ-PSTV $(X, l, \imath)$ is a pair of a marking Q of Δ and a section $\mu\colon \imath(Q) \to H^0(X, \mathcal{M}_X^{\mathrm{gp}})/H^0(X, \mathcal{O}_X^\times)$. A *marked* PSTV over S is a Δ-PSTV with a marking.

(5) For a marking Q of an integral polytope Δ, two markings (Q, μ), (Q, μ') on a log Δ-PSTV are *equivalent* if $\mu \in \mu' H^0(S, \mathcal{M}_S^{\mathrm{gp}}/\mathcal{O}_S^\times)$.

We now explain the log STV over a log point associated to a tropical hypersurface in $N_{\mathbb{R}}$.

Let $\mathcal{V}$ be the tropical hypersurface in $N_{\mathbb{R}}$ defined by $f = \sum_m a_m x^m \in K[M]$ whose Newton polytope is of dimension $r = \operatorname{rank} N$, i.e., $\mathcal{V} = X_{f,\mathrm{trop}}$. Here X_f denotes the hypersurface in $\mathbb{G}_m \otimes N$ defined by $f = 0$. We define the toric variety $\widetilde{X}_f$ over $\mathbb{A}^1$ which is associated to the following fan Σ_f in $N_{\mathbb{R}} \oplus \mathbb{R}$: For each $m \in M$ with $a_m \neq 0$, we define a cone σ_m to be a set

$$\sigma_m := \{(v, l) \in N_{\mathbb{R}} \oplus \mathbb{R}; l \geq 0, \text{ and } \langle m, v\rangle + l v_K(a_m) \leq \langle m', v\rangle + l v_K(a_{m'}) \text{ for all } m' \in M\}.$$

It easily follows that $\{\sigma_m; a_m \neq 0\}$ forms a fan, denoted by Σ_f. By a natural projection $N_{\mathbb{R}} \oplus \mathbb{R} \to \mathbb{R}$, we can consider $\widetilde{X}_f$ as a variety over the toric variety $\mathbb{A}^1$ associated to the ray $\mathbb{R}_{\geq 0}$. We endow $\widetilde{X}_f$ and $\mathbb{A}^1$ with canonical log structures and consider them as fs log schemes, also denoted by $\widetilde{X}_f$ and $\mathbb{A}^1$. It is straightforward to see that the fiber $\widetilde{X}_f \times_{\mathbb{A}^1} O$ of $\widetilde{X}_f$ at the origin O, denoted by X_f, is a log STV over the standard log point whose underlying point is O.

Conversely, we construct a hypersurface from a log STV X over the standard log point S. We first remark that $H^0(X, \mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^{\times})$ is a free $\mathbb{Z}$-module generated by the set of irreducible components $X_1, \ldots, X_u$ of X and generically normal T-orbits of codimension one in X. Let $w_i \in \widetilde{N} := \mathrm{Hom}(\widetilde{M}, \mathbb{Z})$ be the projection $M \to \mathbb{Z}$ corresponding to an irreducible component X_i of X. For each T-orbit Z of X, we define a cone σ_Z in $\widetilde{N}_{\mathbb{R}}$ as the cone over the convex hull of $\{w_i \in \widetilde{N}; Z \cap X_i \neq \emptyset\}$ in $\widetilde{N}_{\mathbb{R}}$. It is easy to see that $\{\sigma_Z; Z \text{ is a } T\text{-orbit}\}$ forms a rational fan in $\widetilde{N}_{\mathbb{R}}$, denoted by Σ_X. We now define the tropical hypersurface $\mathcal{V}_X$ in $N_{\mathbb{R}}$ associated to X as the polyhedral decomposition induced by $N_{\mathbb{R}} \cap \Sigma_X$. Here we identify $N_{\mathbb{R}}$ with its image via the injective homomorphism $N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R}) \to \widetilde{N}_{\mathbb{R}} = \mathrm{Hom}(\widetilde{M}, \mathbb{R})$ induced by the projection $\widetilde{M} \to M$.

Comparing these two definitions of Σ_f and Σ_X, we can conclude the following theorem.

2.3 Theorem *We have a one-to-one correspondence between the set of tropical hypersurfaces in $N_{\mathbb{R}}$ and that of log STVs with characters M.*

The correspondence in the theorem is analogous to one between fans and toric varieties. So we can think of tropical hypersurfaces in $N_{\mathbb{R}}$ as defining polyhedral decompositions of log STVs.

We next discuss a marked or polarized version of the above correspondence.

2.4 Definition (1) For a convex piecewise affine linear function φ on $N_{\mathbb{R}}$, we define the support of φ as the convex hull of $\{\varphi_\sigma - \varphi_\sigma(0); \sigma \text{ is a maximal locus where } \varphi \text{ is an affine linear function } \varphi_\sigma.\}$ in $M_{\mathbb{R}}$.

(2) For a polytope Δ in $M_{\mathbb{R}}$, a *tropical Δ-hypersurface* in $N_{\mathbb{R}}$ is a pair $(\mathcal{V}, \varphi)$ of a tropical hypersurface $\mathcal{V}$ and a convex piecewise affine linear function φ with support Δ that defines $\mathcal{V}$, i.e., $\mathcal{V} := \{v \in N_{\mathbb{R}}; \varphi \text{ is not linear on any neighborhood of } v\}$.

From now on, we consider tropical Δ-hypersurfaces for integral polytopes Δ. We denote by $\Delta_{\mathbb{Z}} := \Delta \cap M$ the set of integral points in Δ.

Let $(\mathcal{V}, \varphi)$ be a tropical Δ-hypersurface in $N_{\mathbb{R}}$. We define the canonical form φ_{can} of φ as a Laurent polynomial $\sum_{m \in \Delta_{\mathbb{Z}}} t^{l_m} x^m$. Here $l_m := \min\{l \in \mathbb{Z}; (m, l) \in P_\varphi\} - \min\{l \in \mathbb{Z}; (m', l) \in P_\varphi, m' \in \Delta_{\mathbb{Z}}\}$, and

$$P_\varphi := \{(m, k); \langle m, v\rangle + kl \geq l\varphi((1/l)v) \text{ for all } (v, l) \in N_{\mathbb{R}} \oplus \mathbb{R}_{>0}\}$$

in $M_{\mathbb{R}} \oplus \mathbb{R}$. Then we have the log Δ-PSTV associated to $\mathcal{V}$ and the marking $\{m; t^{l_m} x^m$ in the boundary of $P_\varphi\} \to \{t^{l_m} \chi^m\} \subset H^0(X, \mathcal{M}^{\mathrm{gp}}/\mathcal{O}_X^{\times})$.

On the other hand, let Δ be an integral polytope, and X a marked log Δ-PSTV over k. After replacing an equivalent marking of X, we assume that the marking of X is of form $\{t^{a_m}x^m; m \in \Delta_{\mathbb{Z}}, a_m \geq 0$, and $a_{m'} = 0$ for some $m'\}$. Then we have the tropical Δ-hypersurface $(\mathcal{V}_X, \varphi_X)$ associated to X, where $\varphi_X := \max\{\langle m, -\rangle + a_m\cdot; m \in \Delta_{\mathbb{Z}}\}$.

We now state a marking version of the previous theorem.

2.5 Theorem *For an r-dimensional integral polytope Δ in $M_{\mathbb{R}}$, we have a one-to-one correspondence between the set of tropical Δ-hypersurfaces with canonical form in $N_{\mathbb{R}}$ and that of equivalence classes of marked Δ-PSTVs over k.*

Proof. The proof is very straightforward. For a marked log Δ-PSTV X, we can easily show $\varphi_{X,\mathrm{can}} = \sum_{m\in\Delta_{\mathbb{Z}}} t^{a_m}x^m$ in the notation above the theorem. So X is the marked log Δ-PSTV associated to $(\mathcal{V}_X, \varphi_X)$.

On the other hand, for a Δ-tropical hypersurface $(\mathcal{V}, \varphi)$, the canonical form $\varphi_{\mathrm{can}} = \sum_{m\in\Delta_{\mathbb{Z}}} t^{l_m}x^m$ defines a piecewise affine linear function $\widetilde{\varphi}_{\mathrm{can}} := \max\{\langle m, -\rangle + l_m\cdot; m \in \Delta_{\mathbb{Z}}\}$. By definition of canonical forms, we can verify that the tropical Δ-hypersurface $\mathcal{V}$ is the one defined by $\widetilde{\varphi}_{\mathrm{can}}$. So $\mathcal{V}$ is also the tropical Δ-hypersurface associated to the marked log Δ-PSTV $X_{\mathcal{V}}$, and φ gives a marking of $X_{\mathcal{V}}$ as explained above. □

3 Examples

Applying the results in the previous section, we study a certain degenerations of toric pairs in terms of tropical geometry.

A hypersurface V in an r-dimensional algebraic torus $T = \mathbb{G}_m \otimes N$ is naturally considered as an ample divisor D on the projective toric variety X associated to the Newton polytope of a defining equation f of V. So we have a toric pair (X, D).

This toric pair is also naturally considered as a marked log PSTV endowed with canonical log structure and marking induced by the defining equation f of V.

On the other hand, all tropical hypersurfaces in $N_{\mathbb{R}}$ come from hypersurfaces in T by definition. Hence we showed relationships between tropical hypersurfaces and log PSTVs in the previous section.

We now classify log PSTVs, which are some degenerations of toric pairs, by using classification of tropical hypersurfaces This is closely related to one of regular (or coherent) subdivisions of integral polytopes.

We first get the following proposition by Theorem 2.5 and triangulation of an integral polytope.

3.1 Proposition *The number of irreducible components of a log Δ-PSTV over a standard log point is at most $r!$-times the volume of Δ, in other words, the self-intersection number of the ample divisor belonging to Δ.*

Proof. Since the volume of an integral polytope which has no integral points except for its vertices is at least $1/r!$, the number of simplices in a triangulation of Δ is at most $r!$-times the volume of Δ, which is equal to the self-intersection number of an ample divisor belonging to Δ. The theorem in the previous section implies that each polytope of a regular subdivision of Δ corresponds to an irreducible component of the underlying scheme of a log Δ-PSTV. The proposition follows from this correspondence. $\square$

3.2 Corollary *In the 2-dimensional case, the maximal number of irreducible components of a log Δ-PSTV is the self-intersection number of an ample divisor belonging to Δ. Moreover, if a log Δ-PSTV has the maximum number of irreducible components, then every irreducible component is isomorphic to $\mathbb{P}^2$, and its polarization is of degree 1.*

Proof. In the 2-dimensional case, each integral simplex which has no integral points except for its vertices has the volume $1/2$. So the corollary follows from the proposition. The rest of assertions are clear. $\square$

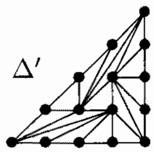
3.3 Example Let k be an algebraically closed field with characteristic 0, and K the field of formal Puiseux series over k. Let Δ_n be the convex hull of $\{(0,0),(n,0),(0,n)\}$ in $M_{\mathbb{R}} = \mathbb{R}^2$ for $n > 0$. The toric variety associated to Δ_n is the projective plane $\mathbb{P}^2$. We consider a certain degenerations of $\mathbb{P}^2$ over K, which we call *tropical degenerations of* $\mathbb{P}^2$.

In the case $n = 1$, a tropical degeneration of $\mathbb{P}^2$ is trivial, i.e., $\mathbb{P}^2$ extends to $\mathbb{P}^2$ over k.

In the case $n = 2$, we can easily see that there exist 7 types of configurations of tropical degenerations of $\mathbb{P}^2$ which corresponds to regular subdivisions of Δ_2 up to the permutations of the vertices. From now on, we consider all subdivisions of Δ_n modulo the permutation of its vertices.

In the case $n = 3$, we have 618 types of configurations of tropical degenerations of $\mathbb{P}^2$ up to projective transformation [6]. We remark that every integral subdivision of Δ_3 is regular [6, Theorem 4.2].

In the case $n = 4$, we have 874854 subdivisions of Δ_4 up to the permutations of vertices. Among them, 873479 subdivisions are regular, and the others are not regular [6, Theorem 5.4]. We can show that every non-regular subdivision is coarser than the maximal subdivision Δ' below.



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