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Arnold multiplicity of divisors on rational homogeneous spaces

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This is a report on my recent work on Arnold multiplicity of divisors on rational homogeneous spaces, and more generally on Fano manifolds. Full details will appear elsewhere.

Let Y be a complex manifold and $y \in Y$ be a point. For an effective divisor D on Y, the Arnold multiplicity $\mu_y(D)$ of D at y is defined by

$$\mu_y(D) := \inf\{m > 0 \ : \ |f|^{-\frac{2}{m}} \text{ is locally integrable at } y.\}$$

where f denotes the local defining function of D at y. This is a very important local invariant of the divisor. Its inverse $\frac{1}{\mu_y(D)}$ is called the log canonical threshold of D at y. When $\dim Y = 1$, $\mu_y(D)$ coincides with the multiplicity $\mathrm{mult}_y(D)$. In general, we have the following inequalities:

$$\frac{\mathrm{mult}_y(D)}{\dim Y} \ \leq \ \mu_y(D) \ \leq \ \mathrm{mult}_y(D).$$

One of the motivation for the current work is the following result of Ein and Lazarsfeld [EL, 3.5].

Theorem 1 *Let (A, Θ) be a principally polarized abelian variety. Then for any positive integer k and $D \in |k\Theta|$,*

$$\mu_x(D) \leq k \text{ for each } x \in A.$$

A principally polarized abelian variety is, among others, a homogeneous projective variety with a natural choice of a line bundle. It is natural to ask whether an analogue of Theorem 1 holds for other homogeneous projective varieties. Our main result is

Theorem 2 *Let G/P be a rational homogeneous space of Picard number 1 and let L be the ample generator of the Picard group of G/P. Then for any positive integer k and $D \in |kL|$,*

$$\mu_x(D) \leq k \text{ for each } x \in G/P.$$

For Grassmannians, Theorem 2 was proved in [Hw]. The proof in [Hw] used Kapranov's work on the derived category of coherent sheaves on Grassmannians and vanishing theorems of Nadel and Demailly. This argument was modelled on that of [EL], which used Mukai's work on the derived category of coherent sheaves on A and Nadel's vanishing theorem. This method seems very hard to generalize to arbitrary G/P. The method we use for Theorem 2 is completely different. It is motivated by another problem. To explain this, let us recall the following well-known conjecture on the multiplicity of pluri-anti-canonical divisors on Fano manifolds.

Conjecture *Let X be a Fano manifold of Picard number 1 and $x \in X$ be a general point. Then for any positive integer k and $D \in |-kK_X|$,*

$$\mathrm{mult}_x(D) \leq 2k \dim X.$$

The bound in Conjecture is optimal and it is achieved when X is a hyperquadric and D is a multiple of a singular hyperplane section. The best known result on Conjecture is essentially the following bound proved in [Ca], [KMM] and [Na].

Theorem 3 *Let X be a Fano manifold of Picard number 1 and $x \in X$ be a general point. Then for any positive integer k and $D \in |-kK_X|$,*

$$\mathrm{mult}_x(D) \leq k(\dim X) \cdot (\dim X + 1).$$

It is natural to ask a similar question for the Arnold multiplicity $\mu_x(D)$. Surprisingly, in this case, one can prove the following optimal result.

Theorem 4 *Let X be a Fano manifold of Picard number 1 and $x \in X$ be a general point. Then for any positive integer k and $D \in |-kK_X|$,*

$$\mu_x(D) \leq k(\dim X + 1).$$

Moreover, if the equality holds for some k and D, then X is the projective space.

Both Theorem 2 and Theorem 4 are simple consequences of Theorem 5 below. Recall that a rational curve C on a compact complex manifold X is free if, under the normalization $f : \mathbf{P}_1 \to C \subset X$, the pull-back f^*T_X of the tangent bundle of X is nef.

Theorem 5 *Let X be a Fano manifold of Picard number 1 and $C \subset X$ be a free rational curve. Then there exists a Zariski dense open subset $U \subset X$ determined by C such that for any effective divisor D on X and $x \in U$,*

$$\mu_x(D) \leq C \cdot D.$$

For Theorem 2, note that on G/P there exists a free rational curve C satisfying $C \cdot L = 1$ (e.g. [Ko, V.1.15]). Thus Theorem 2 follows from Theorem 5 and the homogeneity of X. For Theorem 4, recall from [Ko, IV.2.10] that there exists a free rational curve C satisfying

$C \cdot (-K_X) \leq \dim X + 1$. This gives the inequality in Theorem 4. On the other hand, if X is different from the projective space, [CMS] says that there exists a free rational curve C with $C \cdot (-K_X) \leq \dim X$. Thus the equality in Theorem 4 cannot hold.

Proof of Theorem 5 is motivated by the approach of Nadel [Na] in his proof of Theorem 3. The essential point of Nadel's proof was the following result on the behavior of the multiplicities along a free rational curve, which he called the product theorem.

Theorem 6 *Let X be a compact complex manifold and $C \subset X$ be a free rational curve. For any effective divisor D on X and any two points $x, x' \in C$*

$$| \operatorname{mult}_x(D) - \operatorname{mult}_{x'}(D) | \leq C \cdot D.$$

Motivated by this, we study the behavior of Arnold multiplicities along a free rational curve and get the following.

Theorem 7 *Let X be a compact complex manifold and $C \subset X$ be a free rational curve. For any effective divisor D on X, either $\mu_x(D) \leq C \cdot D$ for each $x \in C$, or $\mu_x(D) = \mu_{x'}(D)$ for any two points $x, x' \in C$.*

To see how Theorem 7 gives Theorem 5, let $\mathrm{RatCurves}^n(X)$ be the space of rational curves on X, following [Ko, II.2]. Let $\mathcal{K}$ be a component of $\mathrm{RatCurves}^n(X)$ to which C belongs. By [Ko, IV.4.14] applied to the family $\mathbf{P}_1 \times \mathcal{K} \to \mathcal{K}$, there exists an open set $W \subset X \times X$ such that if $(x_1, x_2) \in W$, then x_1 and x_2 can be connected by a connected chain of free rational curves belonging to $\mathcal{K}$. Then we choose $U \subset X$ as a Zariski open subset in the image of the projection of W to the first factor. Suppose there exists a point $x_1 \in U$ with $\mu_{x_1}(D) > C \cdot D$. We can choose a point $x_2 \notin D$ such that x_1 and x_2 can be connected by a connected chain of free rational curves belonging to $\mathcal{K}$. Applying Theorem 7 repeatedly, we get $0 = \mu_{x_2}(D) = \mu_{x_1}(D) > C \cdot D$, a contradiction.

The proof of Theorem 7 uses the following Lemma which is a special case of [Vi, Prop. 5.19].

Lemma *Let T be a complex manifold and D be an effective divisor in $T \times \mathbf{P}_1$ such that for a general point $t \in T$, D has intersection number d with the curve $\{t\} \times \mathbf{P}_1$. Then for any $t_o \in T$, either $\mu_x(D) \leq d$ for each $x \in \{t_o\} \times \mathbf{P}_1$, or $\mu_x(D) = \mu_{x'}(D)$ for any two points $x, x' \in \{t_o\} \times \mathbf{P}_1$.*

In fact, let $\mathrm{Hom}^{free}(\mathbf{P}_1, X)$ be the space of free morphisms and $F : \mathbf{P}_1 \times \mathrm{Hom}^{free}(\mathbf{P}_1, X) \to X$ be the evaluation morphism, as in [Ko, II.3.5.4]. By [Ko, II.3.5.4], $\mathrm{Hom}^{free}(\mathbf{P}_1, X)$ is non-singular and F is a smooth morphism. This implies, by [La, 9.5.45], for each $u \in \mathbf{P}_1 \times \mathrm{Hom}^{free}$,

$$\mu_u(F^*D) = \mu_{F(u)}(D).$$

Thus Theorem 7 is a direct consequence of Lemma applied to $T = \mathrm{Hom}^{free}(\mathbf{P}_1, X)$.

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