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G-Constellations and Resolutions of Quotient Singularities

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1 Background

Consider an affine scheme $\mathbb{C}^n = \text{Spec } R$, where by $R$ we denote the ring $\mathbb{C}[x_1, \ldots, x_n]$. By $X$ we denote the quotient space $\mathbb{C}^n/G = \text{Spec } R^G$. By $Y$ we denote a choice of a resolution of $X$.

\[ \begin{array}{ccc}
Y & \xrightarrow{\pi} & \mathbb{C}^n \\
\downarrow & & \downarrow q \\
X & & 
\end{array} \]

The singular quotient space $X$ is in a certain sense ([Muk03], Example 11.8) a coarse moduli space for the set-theoretical orbits of $G$ in $\mathbb{C}^n$. A natural question to ask was whether we can refine a concept of an 'orbit of $G$ in $\mathbb{C}^n$' and state a moduli problem for it which yields a fine moduli space $Y$ which resolves the singularities of $X$.

The first step was to equip an orbit with an appropriate scheme-theoretic structure:

**Definition 1.1.** A $G$-cluster is a $G$-invariant subscheme $Z$ of $\mathbb{C}^n$ of dimension 0 whose ring $\Gamma(Z, \mathcal{O}_Z)$ is a regular representation of $G$.

E.g. any free orbit of $G$ supports a unique $G$-cluster: the reduced induced closed subscheme structure. On the other hand, we find many different $G$-clusters supported at the fixed point orbit at the origin of $\mathbb{C}^n$.

Following the ideas of Nakamura, Reid introduced in [Rei97] the scheme $G$-Hilb, the fine moduli space of all $G$-clusters. It comes equipped with a Hilbert-Chow morphism $G$-Hilb $\mathbb{C}^n \to X$ which sends each $G$-cluster to its set-theoretic support. The main irreducible component of $G$-Hilb $\mathbb{C}^n$ birational to $X$ can be identified (e.g. [IN00], §2) with the scheme $\text{Hilb}^G \mathbb{C}^n$ introduced by Nakamura and Ito in [IN96]. They then proceeded to show that for $G$ a finite subgroup of $\text{SL}_2(\mathbb{C})$, the scheme $\text{Hilb}^G \mathbb{C}^n$ is the unique crepant minimal resolution of $\mathbb{C}^2/G$. 

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Then Nakamura showed by explicit toric geometry computations [Nak00] that for $G$ a finite abelian subgroup of $\text{SL}_3(\mathbb{C})$, the scheme $\text{Hilb}^G \mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$. He conjectured that the same is true for the non-abelian case.

This conjecture was settled by Bridgeland, King and Reid in [BKR01]. They use derived category methods and establish a category equivalence $\mathcal{D}(Y) \to \mathcal{D}^G(\mathbb{C}^n)$ between the bounded derived categories of coherent sheaves on $Y = \text{Hilb}^G \mathbb{C}^n$ and of $G$-equivariant coherent sheaves on $\mathbb{C}^n$, respectively. Under a certain assumption on the dimension of the fibers of $Y$, which holds automatically when $n \leq 3$, they prove that the Fourier-Mukai transform which uses the structure sheaf of the universal $G$-cluster $\mathcal{U}_G \subset Y \times \mathbb{C}^n$ is the requisite equivalence. In particular, this shows that $Y$ is a crepant resolution of $X$, proving Nakamura’s conjecture. It is then further shown ([BKR01], §8) that in the case of $n = 3$, $\text{Hilb}^G \mathbb{C}^3$ is the only component of $G$-$\text{Hilb} \mathbb{C}^3$, i.e. $G$-$\text{Hilb} \mathbb{C}^3$ is connected. In dimension two this was proven by Ishii in [Ish02], while in dimensions four and higher it is known to be false.

For $n \geq 3$ crepant resolutions of $\mathbb{C}^n/G$, if they exist, are not necessarily unique. The question arose whether $G$-clusters can be generalised further, to obtain the other crepant resolutions by a moduli space construction. Subsequent research had shown that it was not necessary to give an orbit a subscheme structure - it is sufficient to equip an orbit with a coherent sheaf that looks like what we would expect of an image of a skyscraper sheaf of a point under a derived category equivalence as above. This generalisation was a concept of a $G$-constellation given by Craw in his thesis [Cra01]:

**Definition 1.2.** A $G$-constellation is a $G$-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$, whose global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$ form a regular representation of $G$.

Note that a priori a definition of $G$-constellation doesn’t exclude sheaves supported at more than one orbit of $G$. However a gnat-family consists only of those supported at a single orbit.

Observe that, tautologically, the structure sheaf of any $G$-cluster is a $G$-constellation. In fact on a free orbit this all we get: the concepts of a $G$-constellation, a $G$-cluster and a set-theoretic orbit coincide where $G$ acts freely. At the origin, however, there are many $G$-constellations which do not arise as structure sheaves of $G$-clusters. Too many in fact: the moduli space of all $G$-constellations is non-separated at the origin, suggesting that some sort of stability conditions are needed.

These came to us courtesy of a natural 1-to-1 correspondence existing between $G$-constellations and representations of the McKay quiver of $G$ into the regular representation of $G$. This allows for the use of an earlier result of King [Kin94] on GIT construction of moduli spaces of quiver representations to introduce the stability conditions known as $\theta$-stability on $G$-constellations and to construct for any given stability condition $\theta$ a moduli space $M_\theta$ of $\theta$-stable $G$-constellations together with a projective morphism to $X$ and a
universal $\theta$-stable $G$-constellation $U_\theta$ in $\text{Coh} Y \times \mathbb{C}^n$. In a quiver-theoretic context, Kronheimer [Kro89] had already considered these moduli spaces and have studied the chamber structure in the space $\Pi$ of stability parameters $\theta$, where all values of $\theta$ in the same chamber yield the same $M_\theta$. The methods of [BKR01] can be then extended to show that, under the same assumptions on the fiber dimensions of $M_\theta$, the Fourier-Mukai transform $D(M_\theta) \to D^G(\mathbb{C}^n)$ is an equivalence of categories, which makes the main irreducible component of $M_\theta$ a crepant resolution of $\mathbb{C}^n/G$. In case of an abelian $G$, an explicit description of this coherent component is provided in toric terms by Craw, Maclagan and Thomas in [CMT05a], [CMT05b].

Craw in his thesis conjectured that when $G$ is a finite subgroup of $\text{SL}_3(\mathbb{C})$ every crepant resolution projective over $\mathbb{C}^3/G$ can be realised as a moduli space $M_\theta$ of $\theta$-stable $G$-constellations for some chamber in $\Pi$. In the case of $G$ being abelian, this was proved by Craw and Ishii in [CI04].

Thus one motivation for the study of families of $G$-constellations on a fixed resolution $Y$ is an observation that, as evident from [CI04], there exist stability parameters $\theta$ for which the GIT construction yields isomorphic moduli spaces $M_\theta$, but equips them with different tautological families of $G$-constellations $U_\theta$. Another is the desire to obtain for a given crepant resolution $Y$ a direct construction of the derived McKay equivalence $D(Y) \sim D^G(\mathbb{C}^n)$ as a Fourier-Mukai functor using an appropriate $G$-constellation family. Finally, the question of a moduli construction of non-projective (over $X$) crepant resolutions still remains open.

2 Gnatt-Families

Rather than constructing a resolution as a moduli space of $G$-constellations, we take an arbitrary (not necessarily projective or crepant) resolution of $X$ and study the flat families of $G$-constellations that it can parametrise.

We would like for a family of $G$-constellations to be a flat $\mathcal{O}_Y$-module, whose restriction to any point of $Y$ would give us the respective $G$-constellation. From this point of view, it would be better to consider, instead of the whole $G$-constellation $\mathcal{F}$, just its space of global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$. It is a vector space $V$ with $G$ and $R$ actions, satisfying

$$g.(f.v) = (g.f).(g.v) \quad (2.1)$$

As $\mathbb{C}^n$ is affine, functor $(\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ recovers $\mathcal{F}$ from $\Gamma(\mathbb{C}^n, \mathcal{F})$, and (2.1) defines the $G$-equivariant structure.

It is convinient to view such vector spaces as modules for the following non-commutative algebra:

**Definition 2.1.** A cross-product algebra $R \rtimes G$ is an algebra, which has the vector space structure of $R \otimes_{\mathbb{C}} \mathbb{C}[G]$ and the product defined by setting,
for all \( g_1, g_2 \in G \) and \( f_1, f_2 \in R, \)

\[
(f_1 \otimes g_1) \times (f_2 \otimes g_2) = (f_1 \cdot g_1) \otimes (g_1 g_2) \tag{2.2}
\]

Functors \( \hat{\bullet} = (\bullet) \otimes_R O_{\mathbb{C}^n} \) and \( \Gamma(\mathbb{C}^n, \bullet) \) give an equivalence between the categories of \( R \rtimes G \)-modules and \( \text{of quasi-coherent } G \)-equivariant sheaves on \( \mathbb{C}^n \).

This is not a pure formalism - \( R \rtimes G \) is one of the non-commutative crepant resolutions of \( \mathbb{C}^n/G \), a certain class of non-commutative algebras introduced by Michel van den Bergh in [dB02] as an analogue of a commutative crepant resolution for an arbitrary non-quotient Gorenstein singularity. For three-dimensional terminal singularities, van den Bergh shows ([dB02], Theorem 6.3.1) that if a non-commutative crepant resolution \( Q \) exists, then it is possible to construct commutative crepant resolutions as moduli spaces of certain stable \( Q \)-modules.

Under \( \Gamma(\mathbb{C}^n, \bullet) \), to \( G \)-constellations correspond \( R \rtimes G \)-modules, which are isomorphic, as representations of \( G \), to the regular representation \( V_{\text{reg}} \). By abuse of notation, we shall use the term \( G \)-constellations to also mean such \( R \rtimes G \)-modules. This interpretation allows us to define a family of \( G \)-constellations as a locally-free sheaf on \( Y \), instead of \( Y \times \mathbb{C}^n \):

**Definition 2.2.** A family of \( G \)-constellations parametrised by \( Y \) is a sheaf \( F \) of \( (R \rtimes G) \otimes_{\mathbb{C}} O_Y \)-modules on \( Y \), locally free as an \( O_Y \)-module, such that, for any point \( \iota_p : p \to Y \), the fiber \( F|_p = \iota_p^*F \) is a \( G \)-constellation.

We wish to develop a notion of a geometrically natural family, in which for any \( p \in Y \) the \( G \)-constellation \( F|_p \) would be geometrically related to the \( G \)-orbit \( q^{-1} \pi(p) \). For example, the \( G \)-constellation \( \tilde{F}|_p \), as a sheaf on \( \mathbb{C}^n \), is supported on a finite union of \( G \)-orbits. We could ask, mimicking the moduli spaces \( M_\theta \) of \( \theta \)-stable \( G \)-constellations and their tautological families, for this support to be precisely \( q^{-1} \pi(p) \).

This turns out to be enough to warranty a much wider range of naturality properties.

**Definition 2.3.** A generically natural family of \( G \)-constellations parametrised by \( Y \) (or a gnat-family, for short) is a family \( F \) of \( G \)-constellations, such that for every \( p \in Y \)

\[
\text{Supp}_{\mathbb{C}^n}(F|_p) = q^{-1} \pi(p)
\]

**Proposition 2.4.** Let \( F \) be a family of \( G \)-constellations parametrised by \( Y \). Then the following are equivalent:

1. On any \( U \subset Y \), such that \( \pi U \) consists of free orbits, \( F \) is equivalent (locally isomorphic) to \( \pi^*q_*O_{\mathbb{C}^n} \).
2. There exists a \((\mathbb{R} \times G) \otimes_{\mathbb{C}} K(Y)\)-module isomorphism:

\[ F_{|p_Y} \sim (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{p_Y} \]

where \(p_Y\) is the generic point of \(Y\).

3. There exists an \((\mathbb{R} \times G) \otimes_{\mathbb{C}} \mathcal{O}_Y\)-module embedding

\[ F \hookrightarrow K(\mathbb{C}^n) \]

where \(\mathcal{O}_Y\)-module structure on \(K(\mathbb{C}^n)\) is induced by the map \(q : Y \rightarrow X\).

4. \(F\) is a gnat-family.

5. The action of \((\mathbb{R} \times G) \otimes_{\mathbb{C}} \mathcal{O}_Y\) on \(F\) descends to the action of \((\mathbb{R} \times G) \otimes_{R^G} \mathcal{O}_Y\), where the \(R^G\)-module structure on \(\mathcal{O}_Y\) is induced by the map \(q : Y \rightarrow X\).

**Sketch.** Implications \(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5\) are quite straightforward. The interesting one is \(5 \Rightarrow 1\).

Consider a natural algebra homomorphism

\[ \Psi : (\mathbb{R} \times G) \otimes_{R^G} \mathcal{O}_Y \rightarrow \text{End}_{\mathcal{O}_Y}(F) \]

LHS is isomorphic to \(\pi^* \text{End}_{\mathcal{O}_X}(q_* \mathcal{O}_{\mathbb{C}^n})\). Over \(U\), as \(q\) is flat over \(U\), LHS is further isomorphic to \(\text{End}_{\mathcal{O}_Y}(\pi^* q_* \mathcal{O}_{\mathbb{C}^n})\). Thus we have

\[ \Psi' : \text{End}_{\mathcal{O}_U}(\pi^* q_* \mathcal{O}_{\mathbb{C}^n}) \rightarrow \text{End}_{\mathcal{O}_U}(F) \]

It is a homomorphism of (split) Azumaya algebras of the same constant rank, which is an isomorphism on the centers. Hence \(\Psi'\) must be an isomorphism itself. Then, by Skolem-Noether theorem, \(\Psi'\) must locally be induced by isomorphisms \(\pi^* q_* \mathcal{O}_{\mathbb{C}^n} \rightarrow F\).

\[ \square \]

3. **\(G\)-divisors**

Since \(G\) is abelian, any family \(F\) of \(G\)-constellations on \(Y\) splits into invertible eigensheaves: \(F = \bigoplus_{X \in G^\vee} F_X\). If \(F\) is also a gnat-family, then it can be embedded into \(K(\mathbb{C}^n)\). Now, generally, on a scheme \(S\) an invertible sheaf embedded into \(K(S)\) defines a Cartier divisor on \(S\).

Therefore, just as the group \(K^*_G(\mathbb{C}^n)^*\) of the invertible \(G\)-homogeneous elements of \(K(\mathbb{C}^n)\) extends \(K^*(Y)\):

\[ 1 \rightarrow K^*(Y) \rightarrow K^*_G(\mathbb{C}^n) \overset{\varphi}{\rightarrow} G^\vee \rightarrow 1 \quad (3.1) \]

we extend the group of Cartier divisors on \(Y\) as follows:

5
**Definition 3.1.** A rational function \( f \in K^*(\mathbb{C}^n) \) is said to be \( G \)-homogeneous (of weight \( \chi \)), if there exists a character \( \chi \in G^\vee \) such that
\[
g \cdot f = \chi(g) f \quad \forall g \in G
\]

**Definition 3.2.** A \( G \)-Cartier divisor on \( Y \) is a global section of the sheaf of multiplicative groups \( K^*_G(\mathbb{C}^n)/\mathcal{O}_Y^* \), where the sheaf \( K^*_G(\mathbb{C}^n) \) is the constant sheaf on \( Y \) of the \( G \)-homogeneous elements of \( K(\mathbb{C}^n) \) and the sheaf \( \mathcal{O}_Y^* \) is the sheaf of invertible regular functions on \( Y \).

Similar to the ordinary Cartier divisors, a \( G \)-Cartier divisor can be specified by a set of pairs \((U_i, f_i)\), where \( U_i \) are an open cover of \( Y \) and \( f_i \) are \( G \)-homogeneous rational functions on \( \mathbb{C}^n \), such that for any \( i \) and \( j \), \( f_i/f_j \) defines an invertible regular function on \( U_i \cap U_j \).

As with ordinary Cartier divisors, we say that a \( G \)-Cartier divisor is principal if it lies in the image of the natural map \( K^*_G(\mathbb{C}^n) \rightarrow K^*_G(\mathbb{C}^n)/\mathcal{O}_Y^* \) and call two divisors linearly equivalent if their difference is principal.

Thus, we obtain a short exact sequence of abelian groups:
\[
1 \rightarrow \text{Car}(Y) \rightarrow G\text{-Car}(Y) \xrightarrow{\rho} G^\vee \rightarrow 1 \quad (3.2)
\]

We call an image of a Cartier divisor \( D \) under the map \( \rho \) its weight and say that \( D \) is a \( \rho(D) \)-Cartier divisor.

The construction of the invertible subsheaf \( \mathcal{L}(D) \) of \( K(Y) \) corresponding to a Cartier divisor \( D \), extends naturally to a construction of an invertible subsheaf \( \mathcal{L}(D) \) of \( K^*_G(\mathbb{C}^n) \) corresponding to a \( G \)-Cartier divisor \( D \).

**Proposition 3.3.** The map \( D \rightarrow \mathcal{L}(D) \) gives an isomorphism between \( G\text{-Car} Y \) and the group of invertible \( G \)-subsheaves of \( K(\mathbb{C}^n) \). Furthermore, it descends to an isomorphism of the group \( G\text{-Cl} \) of \( G \)-Cartier divisors up to linear equivalence and the group \( G\text{-Pic} \) of invertible \( G \)-sheaves on \( Y \).

We now seek to define a matching notion of a \( G \)-Weil divisor. The key notion is: valuations at prime divisors of \( Y \) define a unique group homomorphism \( \text{val}_K \) from \( K^*(Y) \) to \( \text{Div} Y \), the group of Weil divisors. Looking at the short exact sequence (3.1), we see that \( \text{val}_K \) must extend uniquely to a homomorphism \( \text{val}_{K_G} \) from \( K^*_G(\mathbb{C}^n) \) to \( \mathbb{Q}\text{-Div} Y \), as \( G^\vee \) is finite and \( \mathbb{Q} \) is injective. We further obtain a quotient homomorphism \( \text{val}_{G^\vee} \) from \( G^\vee \) to \( \mathbb{Q}/\mathbb{Z}\text{-Div} Y \).

The short exact sequence (3.2) now becomes a commutative diagram:
\[
\begin{array}{ccccccccc}
1 & \rightarrow & \text{Car} Y & \rightarrow & G\text{-Car} Y & \xrightarrow{\rho} & G^\vee & \rightarrow & 1 \\
\text{val}_K & & \downarrow & & \text{val}_{K_G} & & \downarrow & & \text{val}_{G^\vee} & \\
0 & \rightarrow & \text{Div} Y & \rightarrow & \mathbb{Q}\text{-Div} Y & \rightarrow & \mathbb{Q}/\mathbb{Z}\text{-Div} Y & \rightarrow & 0
\end{array}
\quad (3.3)
\]
Aiming to have a short exact sequence similar to (3.2), we now define the group $G\text{-Div}_Y$ of $G$-Weil divisors to be the subgroup of $\mathbb{Q}\text{-Div}_Y$, which consists of the pre-images of $\text{val}_{G^\vee}(G^\vee) \subset \mathbb{Q}/\mathbb{Z}\text{-Div}_Y$.

We call a $G$-Weil divisor principal if it is an image of a single function $f \in K^*_G(\mathbb{C}^n)$ under $\text{val}_{K^0}$, call two $G$-Weil divisors linearly equivalent if their difference is principal and call a divisor $\sum q_iD_i$ effective if all $q_i \geq 0$.

We now have a following commutative diagram:

$$
\begin{array}{c}
1 & \longrightarrow & \text{Car}_Y & \longrightarrow & G\text{-Car}_Y & \longrightarrow & G^\vee & \longrightarrow & 1 \\
\text{val}_K & \downarrow & \text{val}_{K^0} & \downarrow & \text{val}_{G^\vee} & \\
0 & \longrightarrow & \text{Div}_Y & \longrightarrow & G\text{-Div}_Y & \longrightarrow & \text{val}_{G^\vee}(G^\vee) & \longrightarrow & 0
\end{array}
$$

(3.4)

A priori there is no reason for $\text{val}_{K^0}$ in (3.4) to be an isomorphism. Indeed, although all the definitions above make sense for a general scheme $Y$ birational to $X$, simply assuming $Y$ to be smooth is not enough to warranty $G$-Cartier and $G$-Weil divisors to be isomorphic or even well-behaved. For an example let $Y$ be the smooth locus of $X$. It can be shown, that while $\text{val}_K$ is an isomorphism, $\text{val}_{K^0}$ is not even injective as $G\text{-Car}_Y$ has torsion. And $\text{val}_{G^\vee}$ is the zero map, thus $G\text{-Div}_Y$ is just $\text{Div}_Y$.

**Proposition 3.4.** If $Y$ is smooth and proper over $X$, then $\text{val}_K$, $\text{val}_{K^0}$ and $\text{val}_{G^\vee}$ in (3.4) are all isomorphisms.

## 4 Classification of the gnat-Families

Given a gnat-family $\mathcal{F} = \oplus \mathcal{F}_X$, we can embed it into $K(\mathbb{C}^n)$. An image of $\mathcal{F}_X$ under such an embedding is an invertible subsheaf of $K^*_G(\mathbb{C}^n)$ and therefore the embedding defines a unique $G$-Weil divisor set $\{D_X\}_{X \in G^\vee}$ on $Y$ such that the image of $\mathcal{F}$ in $K(\mathbb{C}^n)$ is $\oplus \mathcal{L}(-D_X)$.

Conversely, given a $G$-divisor set $\{D_X\}_{X \in G^\vee}$ such that each $D_X$ is a $\chi$-Weil divisor, we could ask when is $\oplus \mathcal{L}(-D_X)$ a gnat-family.

**Proposition 4.1.** Let $\{D_X\}_{X \in G^\vee}$ be as above. Then $\oplus \mathcal{L}(-D_X)$ is a gnat-family if and only if for any $G$-homogeneous $f \in R$ and any $X \in G^\vee$ we have

$$
D_X + (f) - \rho(f) \geq 0
$$

(4.1)

where $\rho(f) \in G^\vee$ is the weight of $f$.

**NB:** Observe that the condition (4.1) is equivalent to a set of $|G|$ inequalities for each prime Weil divisor, and that these sets are all independent of each other.

We call $G$-divisor sets $\{D_X\}_{X \in G^\vee}$ which satisfy (4.1) the **reductor sets**.

Recall that in moduli problems it is a standard practice to consider the families up to equivalence, that is up to a local isomorphism.
Theorem 4.1. The isomorphism classes of gnat-families on $Y$ are in a one-to-one correspondence with the linear equivalence classes of the reductor sets $\{D_x\}$. The equivalence classes of gnat-families on $Y$ are in a one-to-one correspondence with the reductor sets $\{D_x\}$, in which $D_{x_0} = 0$.

We say that a reductor set $\{D_x\}$ is normalised, if $D_{x_0} = 0$.

Proposition 4.2 (Canonical family). Define the divisor set $\{D_x\}$ by

$$D_x = \sum_P v(P, \chi)P$$

Then $\{D_x\}$ is a normalised reductor set. Moreover, the corresponding family $\oplus \mathcal{L}(-D_x)$ is the pushdown to $Y$ of the structure sheaf of the normalization of the reduced fibre product $Y \times_X \mathbb{C}^n$.

Proposition 4.3 (Maximal shift family). Define the divisor set $\{M_x\}$ by

$$M_x = \sum_P \min_{f \in \mathbb{R}_x} v_P(f)P$$

Then $\{M_x\}$ is a normalised reductor set.

NB: It can be shown that, for any $\chi \in G^\vee$, the coefficient of $M_x$ at a prime Weil divisor $P$ is non-zero if and only if $P$ is exceptional or the image of $P$ in $X$ is the branch divisor of the quotient map $\mathbb{C}^n \to X$. Therefore, for each $\chi \in G^\vee$, the coefficient of $M_x$ is non-zero at only finitely many prime divisors in $Y$.

Proposition 4.4. Let $\{D_x\}$ be any normalised reductor set. Then

$$-M_{x-1} \leq D_x \leq M_x$$

for any $\chi \in G^\vee$.

Corollary 4.5. The number of equivalence classes of gnat-families is finite.

We summarise our results in the following theorem:

Theorem 4.2 (Classification of gnat-families). Let $G$ be a finite abelian subgroup of $GL_n(\mathbb{C})$, $X$ the quotient of $\mathbb{C}^n$ by the action of $G$, $Y$ nonsingular and $\pi : Y \to X$ a proper birational map. Then isomorphism classes of gnat-families on $Y$ are in 1-to-1 correspondence with linear equivalence classes of $G$-divisor sets $\{D_x\}_{\chi \in G^\vee}$, each $D_x$ a $\chi$-Weil divisor, which satisfy the inequalities

$$D_x + (f) - D_{\chi \rho(f)} \geq 0 \quad \forall \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Such a divisor set $\{D_x\}$ corresponds then to a gnat-family $\bigoplus \mathcal{L}(-D_x)$.
This correspondence descends to a 1-to-1 correspondence between equivalence classes of gnat-families and sets \( \{D_x\} \) as above and with \( D_{x_0} = 0 \). Furthermore, each divisor \( D_x \) in such a set satisfies inequality
\[
-M_{x^{-1}} \leq D_x \leq M_x
\]
where \( \{M_x\} \) is a fixed divisor set defined by
\[
M_x = \sum_{P} (\min_{f \in R_x} v_P(f)) P
\]
As a consequence, the number of equivalence classes of gnat-families is finite.

References


