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G-Constellations and Resolutions of Quotient Singularities

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1 Background

Consider an affine scheme $\mathbb{C}^n = \text{Spec } R$, where by $R$ we denote the ring $\mathbb{C}[x_1, \ldots, x_n]$. By $X$ we denote the quotient space $\mathbb{C}^n/G = \text{Spec } R^G$. By $Y$ we denote a choice of a resolution of $X$.

$$Y \xrightarrow{\pi} \mathbb{C}^n \xrightarrow{q} X$$

The singular quotient space $X$ is in a certain sense ([Muk03], Example 11.8) a coarse moduli space for the set-theoretical orbits of $G$ in $\mathbb{C}^n$. A natural question to ask was whether we can refine a concept of an 'orbit of $G$ in $\mathbb{C}^n$' and state a moduli problem for it which yields a fine moduli space $Y$ which resolves the singularities of $X$.

The first step was to equip an orbit with an appropriate scheme-theoretic structure:

**Definition 1.1.** A $G$-cluster is a $G$-invariant subscheme $Z$ of $\mathbb{C}^n$ of dimension 0 whose ring $\Gamma(Z, \mathcal{O}_Z)$ is a regular representation of $G$.

E.g. any free orbit of $G$ supports a unique $G$-cluster: the reduced induced closed subscheme structure. On the other hand, we find many different $G$-clusters supported at the fixed point orbit at the origin of $\mathbb{C}^n$.

Following the ideas of Nakamura, Reid introduced in [Rei97] the scheme $G$-Hilb, the fine moduli space of all $G$-clusters. It comes equipped with a Hilbert-Chow morphism $G$-Hilb $\mathbb{C}^n \rightarrow X$ which sends each $G$-cluster to its set-theoretic support. The main irreducible component of $G$-Hilb $\mathbb{C}^n$ birational to $X$ can be identified (e.g. [IN00], §2) with the scheme $\text{Hilb}^G \mathbb{C}^n$ introduced by Nakamura and Ito in [IN96]. They then proceeded to show that for $G$ a finite subgroup of $\text{SL}_2(\mathbb{C})$, the scheme $\text{Hilb}^G \mathbb{C}^n$ is the unique crepant minimal resolution of $\mathbb{C}^2/G$. 
Then Nakamura showed by explicit toric geometry computations [Nak00] that for $G$ a finite abelian subgroup of $\text{SL}_3(\mathbb{C})$, the scheme $\text{Hilb}^G \mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$. He conjectured that the same is true for the non-abelian case.

This conjecture was settled by Bridgeland, King and Reid in [BKR01]. They use derived category methods and establish a category equivalence $D(Y) \rightarrow D^G(\mathbb{C}^n)$ between the bounded derived categories of coherent sheaves on $Y = \text{Hilb}^G \mathbb{C}^n$ and of $G$-equivariant coherent sheaves on $\mathbb{C}^n$, respectively. Under a certain assumption on the dimension of the fibers of $Y$, which holds automatically when $n \leq 3$, they prove that the Fourier-Mukai transform which uses the structure sheaf of the universal $G$-cluster $U_G \subset Y \times \mathbb{C}^n$ is the requisite equivalence. In particular, this shows that $Y$ is a crepant resolution of $X$, proving Nakamura’s conjecture. It is then further shown ([BKR01], §8) that in the case of $n = 3$, $\text{Hilb}^G \mathbb{C}^3$ is the only component of $G$-$\text{Hilb} \mathbb{C}^3$, i.e. $G$-$\text{Hilb} \mathbb{C}^3$ is connected. In dimension two this was proven by Ishii in [Ish02], while in dimensions four and higher it is known to be false.

For $n \geq 3$ crepant resolutions of $\mathbb{C}^n / G$, if they exist, are not necessarily unique. The question arose whether $G$-clusters can be generalised further, to obtain the other crepant resolutions by a moduli space construction. Subsequent research had shown that it was not necessary to give an orbit a subscheme structure - it is sufficient to equip an orbit with a coherent sheaf that looks like what we would expect of an image of a skyscraper sheaf of a point under a derived category equivalence as above. This generalisation was a concept of a $G$-constellation given by Craw in his thesis [Cra01]:

**Definition 1.2.** A $G$-constellation is a $G$-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$, whose global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$ form a regular representation of $G$.

Note that a priori a definition of $G$-constellation doesn’t exclude sheaves supported at more than one orbit of $G$. However a gnat-family consists only of those supported at a single orbit.

Observe that, tautologically, the structure sheaf of any $G$-cluster is a $G$-constellation. In fact on a free orbit this all we get: the concepts of a $G$-constellation, a $G$-cluster and a set-theoretic orbit coincide where $G$ acts freely. At the origin, however, there are many $G$-constellations which do not arise as structure sheaves of $G$-clusters. Too many in fact: the moduli space of all $G$-constellations is non-separated at the origin, suggesting that some sort of stability conditions are needed.

These came to us courtesy of a natural 1-to-1 correspondence existing between $G$-constellations and representations of the McKay quiver of $G$ into the regular representation of $G$. This allows for the use of an earlier result of King [Kin94] on GIT construction of moduli spaces of quiver representations to introduce the stability conditions known as $\theta$-stability on $G$-constellations and to construct for any given stability condition $\theta$ a moduli space $M_\theta$ of $\theta$-stable $G$-constellations together with a projective morphism to $X$. and a
universal $\theta$-stable $G$-constellation $U_\theta$ in $\text{Coh} Y \times \mathbb{C}^n$. In a quiver-theoretic context, Kronheimer [Kro89] had already considered these moduli spaces and have studied the chamber structure in the space $\Pi$ of stability parameters $\theta$, where all values of $\theta$ in the same chamber yield the same $M_\theta$. The methods of [BKR01] can be then extended to show that, under the same assumptions on the fiber dimensions of $M_\theta$, the Fourier-Mukai transform $D(M_\theta) \to D^G(\mathbb{C}^n)$ is an equivalence of categories, which makes the main irreducible component of $M_\theta$ a crepant resolution of $\mathbb{C}^n/G$. In case of an abelian $G$, an explicit description of this coherent component is provided in toric terms by Craw, Maclagan and Thomas in [CMT05a], [CMT05b].

Craw in his thesis conjectured that when $G$ is a finite subgroup of $\text{SL}_3(\mathbb{C})$ every crepant resolution projective over $\mathbb{C}^3/G$ can be realised as a moduli space $M_\theta$ of $\theta$-stable $G$-constellations for some chamber in $\Pi$. In the case of $G$ being abelian, this was proved by Craw and Ishii in [CI04].

Thus one motivation for the study of families of $G$-constellations on a fixed resolution $Y$ is an observation that, as evident from [CI04], there exist stability parameters $\theta$ for which the GIT construction yields isomorphic moduli spaces $M_\theta$, but equips them with different tautological families of $G$-constellations $U_\theta$. Another is the desire to obtain for a given crepant resolution $Y$ a direct construction of the derived McKay equivalence $D(Y) \sim D^G(\mathbb{C}^n)$ as a Fourier-Mukai functor using an appropriate $G$-constellation family. Finally, the question of a moduli construction of non-projective (over $X$) crepant resolutions still remains open.

2 Gnat-Families

Rather than constructing a resolution as a moduli space of $G$-constellations, we take an arbitrary (not necessarily projective or crepant) resolution of $X$ and study the flat families of $G$-constellations that it can parametrise.

We would like for a family of $G$-constellations to be a flat $\mathcal{O}_Y$-module, whose restriction to any point of $Y$ would give us the respective $G$-constellation. From this point of view, it would be better to consider, instead of the whole $G$-constellation $\mathcal{F}$, just its space of global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$. It is a vector space $V$ with $G$ and $R$ actions, satisfying

$$g.(f.v) = (g.f).(g.v) \quad (2.1)$$

As $\mathbb{C}^n$ is affine, functor $(\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ recovers $\mathcal{F}$ from $\Gamma(\mathbb{C}^n, \mathcal{F})$, and (2.1) defines the $G$-equivariant structure.

It is convinient to view such vector spaces as modules for the following non-commutative algebra:

**Definition 2.1.** A cross-product algebra $R \rtimes G$ is an algebra, which has the vector space structure of $R \otimes_{\mathbb{C}} \mathbb{C}[G]$ and the product defined by setting,
for all $g_1, g_2 \in G$ and $f_1, f_2 \in R$,
\[(f_1 \otimes g_1) \times (f_2 \otimes g_2) = (f_1 (g_1 f_2)) \otimes (g_1 g_2) \tag{2.2}\]

Functors $\tilde{*} = (\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ and $\Gamma(\mathbb{C}^n, \bullet)$ give an equivalence between the categories of $R \rtimes G$-modules and of quasi-coherent $G$-equivariant sheaves on $\mathbb{C}^n$.

This is not a pure formalism - $R \rtimes G$ is one of the non-commutative crepant resolutions of $\mathbb{C}^n/G$, a certain class of non-commutative algebras introduced by Michel van den Bergh in [dB02] as an analogue of a commutative crepant resolution for an arbitrary non-quotient Gorenstein singularity. For three-dimensional terminal singularities, van den Bergh shows ([dB02], Theorem 6.3.1) that if a non-commutative crepant resolution $Q$ exists, then it is possible to construct commutative crepant resolutions as moduli spaces of certain stable $Q$-modules.

Under $\Gamma(\mathbb{C}^n, \bullet)$, to $G$-constellations correspond $R \rtimes G$-modules, which are isomorphic, as representations of $G$, to the regular representation $V_{\text{reg}}$. By abuse of notation, we shall use the term $G$-constellations to also mean such $R \rtimes G$-modules. This interpretation allows us to define a family of $G$-constellations as a locally-free sheaf on $Y$, instead of $Y \times \mathbb{C}^n$:

**Definition 2.2.** A family of $G$-constellations parametrised by $Y$ is a sheaf $\mathcal{F}$ of $(R \rtimes G) \otimes \mathcal{O}_Y$-modules on $Y$, locally free as an $\mathcal{O}_Y$-module, such that, for any point $\nu : p \to Y$, the fiber $\mathcal{F}|_p = \nu^* \mathcal{F}$ is a $G$-constellation.

We wish to develop a notion of a geometrically natural family, in which for any $p \in Y$ the $G$-constellation $\mathcal{F}|_p$ would be geometrically related to the $G$-orbit $q^{-1}(p)$ . For example, the $G$-constellation $\mathcal{F}|_p$, as a sheaf on $\mathbb{C}^n$, is supported on a finite union of $G$-orbits. We could ask, mimicking the moduli spaces $M_\theta$ of $\theta$-stable $G$-constellations and their tautological families, for this support to be precisely $q^{-1}(p)$.

This turns out to be enough to warranty a much wider range of naturality properties.

**Definition 2.3.** A generically natural family of $G$-constellations parametrised by $Y$ (or a gnat-family, for short) is a family $\mathcal{F}$ of $G$-constellations, such that for every $p \in Y$
\[\text{Supp}_{\mathbb{C}^n}(\mathcal{F}|_p) = q^{-1}(p)\]

**Proposition 2.4.** Let $\mathcal{F}$ be a family of $G$-constellations parametrised by $Y$.
Then the following are equivalent:

1. On any $U \subset Y$, such that $\pi U$ consists of free orbits, $\mathcal{F}$ is equivalent (locally isomorphic) to $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$.
2. There exists a \((R \rtimes G) \otimes \mathbb{C} K(Y)\)-module isomorphism:
\[
\mathcal{F}|_{p_Y} \sim (\pi^*q_*\mathcal{O}_{\mathbb{C}^n})_{p_Y}
\]
where \(p_Y\) is the generic point of \(Y\).

3. There exists an \((R \rtimes G) \otimes \mathcal{O}_Y\)-module embedding
\[
F \hookrightarrow K(\mathbb{C}^n)
\]
where \(\mathcal{O}_Y\)-module structure on \(K(\mathbb{C}^n)\) is induced by the map \(q : Y \rightarrow X\).

4. \(\mathcal{F}\) is a gnat-family.

5. The action of \((R \rtimes G) \otimes \mathcal{O}_Y\) on \(\mathcal{F}\) descends to the action of \((R \rtimes G) \otimes R^G \mathcal{O}_Y\),
where the \(R^G\)-module structure on \(\mathcal{O}_Y\) is induced by the map \(q : Y \rightarrow X\).

Sketch. Implications 1 \(\Rightarrow\) 2 \(\Rightarrow\) 3 \(\Rightarrow\) 4 \(\Rightarrow\) 5 are quite straightforward. The interesting one is 5 \(\Rightarrow\) 1.

Consider a natural algebra homomorphism
\[
\Psi : (R \rtimes G) \otimes R^G \mathcal{O}_Y \rightarrow \text{End}_{\mathcal{O}_Y}(\mathcal{F})
\]
LHS is isomorphic to \(\pi^*\text{End}_X(q_*\mathcal{O}_{\mathbb{C}^n})\). Over \(U\), as \(q\) is flat over \(\pi U\), LHS is further isomorphic to \(\text{End}_{\mathcal{O}_Y}(\pi^*q_*\mathcal{O}_{\mathbb{C}^n})\). Thus we have
\[
\Psi' : \text{End}_{\mathcal{O}_U}(\pi^*q_*\mathcal{O}_{\mathbb{C}^n}) \rightarrow \text{End}_{\mathcal{O}_U}(\mathcal{F})
\]
It is a homomorphism of (split) Azumaya algebras of the same constant rank, which is an isomorphism on the centers. Hence \(\Psi'\) must be an isomorphism itself. Then, by Skolem-Noether theorem, \(\Psi'\) must locally be induced by isomorphisms \(\pi^*q_*\mathcal{O}_{\mathbb{C}^n} \rightarrow \mathcal{F}\).

\[\square\]

3 \hspace{1em} \textit{G-divisors}

Since \(G\) is abelian, any family \(\mathcal{F}\) of \(G\)-constellations on \(Y\) splits into invertible eigensheaves: \(\mathcal{F} = \oplus_{\chi \in G^\vee} \mathcal{F}_\chi\). If \(\mathcal{F}\) is also a gnat-family, then it can be embedded into \(K(\mathbb{C}^n)\). Now, generally, on a scheme \(S\) an invertible sheaf embedded into \(K(S)\) defines a Cartier divisor on \(S\).

Therefore, just as the group \(K^*_G(\mathbb{C}^n)^*\) of the invertible \(G\)-homogeneous elements of \(K(\mathbb{C}^n)\) extends \(K^*(Y)\):
\[
1 \rightarrow K^*(Y) \rightarrow K^*_G(\mathbb{C}^n) \xrightarrow{\delta} G^\vee \rightarrow 1 \tag{3.1}
\]
we extend the group of Cartier divisors on \(Y\) as follows:
**Definition 3.1.** A rational function $f \in K^*({\mathbb{C}}^n)$ is said to be $G$-homogeneous (of weight $\chi$), if there exists a character $\chi \in G^\vee$ such that

$$g \cdot f = \chi(g)f \quad \forall \ g \in G$$

**Definition 3.2.** A $G$-Cartier divisor on $Y$ is a global section of the sheaf of multiplicative groups $K_G^*({\mathbb{C}}^n)/\mathcal{O}_Y^*$, where the sheaf $K_G^*({\mathbb{C}}^n)$ is the constant sheaf on $Y$ of the $G$-homogeneous elements of $K({\mathbb{C}}^n)$ and the sheaf $\mathcal{O}_Y^*$ is the sheaf of invertible regular functions on $Y$.

Similar to the ordinary Cartier divisors, a $G$-Cartier divisor can be specified by a set of pairs $(U_i, f_i)$, where $U_i$ are an open cover of $Y$ and $f_i$ are $G$-homogenous rational functions on $\mathbb{C}^n$, such that for any $i$ and $j$, $f_i/f_j$ defines an invertible regular function on $U_i \cap U_j$.

As with ordinary Cartier divisors, we say that a $G$-Cartier divisor is principal if it lies in the image of the natural map $K^*_G(\mathbb{C}^n) \rightarrow K^*_G(\mathbb{C}^n)/\mathcal{O}_Y^*$ and call two divisors linearly equivalent if their difference is principal.

Thus, we obtain a short exact sequence of abelian groups:

$$1 \rightarrow \text{Car}(Y) \rightarrow G\text{-Car}(Y) \xrightarrow{\rho} G^\vee \rightarrow 1 \quad (3.2)$$

We call an image of a Cartier divisor $D$ under the map $\rho$ its weight and say that $D$ is a $\rho(D)$-Cartier divisor.

The construction of the invertible subsheaf $\mathcal{L}(D)$ of $K(Y)$ corresponding to a Cartier divisor $D$, extends naturally to a construction of an invertible subsheaf $\mathcal{L}(D)$ of $K_G^*({\mathbb{C}}^n)$ corresponding to a $G$-Cartier divisor $D$.

**Proposition 3.3.** The map $D \rightarrow \mathcal{L}(D)$ gives an isomorphism between $G$-Car $Y$ and the group of invertible $G$-subsheaves of $K(\mathbb{C}^n)$. Furthermore, it descends to an isomorphism of the group $G$-Cl of $G$-Cartier divisors up to linear equivalence and the group $G$-Pic of invertible $G$-sheaves on $Y$.

We now seek to define a matching notion of a $G$-Weil divisor. The key notion is: valuations at prime divisors of $Y$ define a unique group homomorphism $\text{val}_K$ from $K^*(Y)$ to $\text{Div} Y$, the group of Weil divisors. Looking at the short exact sequence (3.1), we see that $\text{val}_K$ must extend uniquely to a homomorphism $\text{val}_{K_G}$ from $K^*_G(\mathbb{C}^n)$ to $\mathbb{Q}$-$\text{Div} Y$, as $G^\vee$ is finite and $\mathbb{Q}$ is injective. We further obtain a quotient homomorphism $\text{val}_{G^\vee}$ from $G^\vee$ to $\mathbb{Q}/\mathbb{Z}$-$\text{Div} Y$.

The short exact sequence (3.2) now becomes a commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \rightarrow & \text{Car} Y & \rightarrow & G\text{-Car} Y & \xrightarrow{\rho} & G^\vee & \rightarrow & 1 \\
\text{val}_K & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Div} Y & \rightarrow & \mathbb{Q}\text{-Div} Y & \rightarrow & \mathbb{Q}/\mathbb{Z}\text{-Div} Y & \rightarrow & 0
\end{array}
$$

(3.3)
Aiming to have a short exact sequence similar to (3.2), we now define the group $G$-$\text{Div}_Y$ of $G$-Weil divisors to be the subgroup of $\mathbb{Q}$-$\text{Div}_Y$, which consists of the pre-images of $\text{val}_{G^\vee}(G^\vee) \subset \mathbb{Q}/\mathbb{Z}$-$\text{Div}_Y$.

We call a $G$-Weil divisor principal if it is an image of a single function $f \in K^*_G(\mathbb{C}^n)$ under $\text{val}_{K^0}$, call two $G$-Weil divisors linearly equivalent if their difference is principal and call a divisor $\sum q_i D_i$ effective if all $q_i \geq 0$.

We now have a following commutative diagram:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \text{Car}_Y & \longrightarrow & G$-\text{Car}_Y & \overset{\rho}{\longrightarrow} & G^\vee & \longrightarrow & 1 \\
& \Big\downarrow \text{val}_K & & \Big\downarrow \text{val}_{K^G} & & \Big\downarrow \text{val}_{G^\vee} & & \Big\downarrow & \\
0 & \longrightarrow & \text{Div}_Y & \longrightarrow & G$-\text{Div}_Y & \longrightarrow & \text{val}_{G^\vee}(G^\vee) & \longrightarrow & 0
\end{array}
$$

(3.4)

A priori there is no reason for $\text{val}_{K^G}$ in (3.4) to be an isomorphism. Indeed, although all the definitions above make sense for a general scheme $Y$ birational to $X$, simply assuming $Y$ to be smooth is not enough to warrant $G$-Cartier and $G$-Weil divisors to be isomorphic or even well-behaved. For an example let $Y$ be the smooth locus of $X$. It can be shown, that while $\text{val}_K$ is an isomorphism, $\text{val}_{K^G}$ is not even injective as $G$-$\text{Car}_Y$ has torsion. And $\text{val}_{G^\vee}$ is the zero map, thus $G$-$\text{Div}_Y$ is just $\text{Div}_Y$.

**Proposition 3.4.** If $Y$ is smooth and proper over $X$, then $\text{val}_K$, $\text{val}_{K^G}$ and $\text{val}_{G^\vee}$ in (3.4) are all isomorphisms.

4 Classification of the gnat-Families

Given a gnat-family $\mathcal{F} = \oplus \mathcal{F}_X$, we can embed it into $K(\mathbb{C}^n)$. An image of $\mathcal{F}_X$ under such an embedding is an invertible subsheaf of $K^*_G(\mathbb{C}^n)$ and therefore the embedding defines a unique $G$-Weil divisor set $\{D_X\}_{X \in G^\vee}$ on $Y$ such that the image of $\mathcal{F}$ in $K(\mathbb{C}^n)$ is $\oplus \mathcal{L}(-D_X)$.

Conversely, given a $G$-divisor set $\{D_X\}_{X \in G^\vee}$ such that each $D_X$ is a $\chi$-Weil divisor, we could ask when is $\oplus \mathcal{L}(-D_X)$ a gnat-family.

**Proposition 4.1.** Let $\{D_X\}_{X \in G^\vee}$ be as above. Then $\oplus \mathcal{L}(-D_X)$ is a gnat-family if and only if for any $G$-homogeneous $f \in R$ and any $X \in G^\vee$ we have

$$
D_X + (f) - D_{\chi^p(f)} \geq 0
$$

(4.1)

where $\rho(f) \in G^\vee$ is the weight of $f$.

**NB:** Observe that the condition (4.1) is equivalent to a set of $|G|$ inequalities for each prime Weil divisor, and that these sets are all independent of each other.

We call $G$-divisor sets $\{D_X\}_{X \in G^\vee}$ which satisfy (4.1) the reductor sets.

Recall that in moduli problems it is a standard practice to consider the families up to equivalence, that is up to a local isomorphism.

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Theorem 4.1. The isomorphism classes of gnat-families on $Y$ are in a one-to-one correspondence with the linear equivalence classes of the reductor sets \{D_x\}. The equivalence classes of gnat-families on $Y$ are in a one-to-one correspondence with the reductor sets \{D_x\}, in which $D_{\chi_0} = 0$.

We say that a reductor set \{D_x\} is normalised, if $D_{\chi_0} = 0$.

Proposition 4.2 (Canonical family). Define the divisor set \{D_x\} by

$$D_x = \sum_P v(P, \chi) P$$

Then \{D_x\} is a normalised reductor set. Moreover, the corresponding family $\oplus \mathcal{L}(-D_x)$ is the pushdown to $Y$ of the structure sheaf of the normalization of the reduced fibre product $Y \times_X \mathbb{C}^n$.

Proposition 4.3 (Maximal shift family). Define the divisor set \{M_x\} by

$$M_x = \sum_P \min_{f \in R_x} v_P(f) P$$

Then \{M_x\} is a normalised reductor set.

NB: It can be shown that, for any $\chi \in G^\vee$, the coefficient of $M_x$ at a prime Weil divisor $P$ is non-zero if and only if $P$ is exceptional or the image of $P$ in $X$ is the branch divisor of the quotient map $\mathbb{C}^n \to X$. Therefore, for each $\chi \in G^\vee$, the coefficient of $M_x$ is non-zero at only finitely many prime divisors in $Y$.

Proposition 4.4. Let \{D_x\} be any normalised reductor set. Then

$$-M_{\chi^{-1}} \leq D_x \leq M_\chi$$

for any $\chi \in G^\vee$.

Corollary 4.5. The number of equivalence classes of gnat-families is finite.

We summarise our results in the following theorem:

Theorem 4.2 (Classification of gnat-families). Let $G$ be a finite abelian subgroup of $\text{GL}_n(\mathbb{C})$, $X$ the quotient of $\mathbb{C}^n$ by the action of $G$, $Y$ nonsingular and $\pi : Y \to X$ a proper birational map. Then isomorphism classes of gnat-families on $Y$ are in 1-to-1 correspondence with linear equivalence classes of $G$-divisor sets \{D_x\}_{\chi \in G^\vee}, each $D_x$ a $\chi$-Weil divisor, which satisfy the inequalities

$$D_x + (f) - D_{\chi \rho(f)} \geq 0 \quad \forall \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Such a divisor set \{D_x\} corresponds then to a gnat-family $\oplus \mathcal{L}(-D_x)$.
This correspondence descends to a 1-to-1 correspondence between equivalence classes of gnat-families and sets \( \{D_x\} \) as above and with \( D_{x_0} = 0 \). Furthermore, each divisor \( D_x \) in such a set satisfies inequality

\[-M_x^{-1} \leq D_x \leq M_x\]

where \( \{M_x\} \) is a fixed divisor set defined by

\[M_x = \sum_{P} (\min_{f \in R_x} v_P(f))P\]

As a consequence, the number of equivalence classes of gnat-families is finite.

References


