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G-Constellations and Resolutions of Quotient Singularities

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1 Background

Consider an affine scheme $\mathbb{C}^n = \text{Spec } R$, where by R we denote the ring $\mathbb{C}[x_1, \ldots, x_n]$. By X we denote the quotient space $\mathbb{C}^n/G = \text{Spec } R^G$. By Y we denote a choice of a resolution of X.

$$\begin{tikzcd} Y \arrow[dr, "\pi"] & & \mathbb{C}^n \arrow[dl, "q"'] \\ & X & \end{tikzcd}$$

The singular quotient space X is in a certain sense ([Muk03], Example 11.8) a coarse moduli space for the set-theoretical orbits of G in $\mathbb{C}^n$. A natural question to ask was whether we can refine a concept of an 'orbit of G in $\mathbb{C}^n$' and state a moduli problem for it which yields a fine moduli space Y which resolves the singularities of X.

The first step was to equip an orbit with an appropriate scheme-theoretic structure:

Definition 1.1. A G-`cluster` is a G-invariant subscheme $\mathcal{Z}$ of $\mathbb{C}^n$ of dimension 0 whose ring $\Gamma(Z, \mathcal{O}_Z)$ is a regular representation of G.

E.g. any free orbit of G supports a unique G-cluster: the reduced induced closed subscheme structure. On the other hand, we find many different G-clusters supported at the fixed point orbit at the origin of $\mathbb{C}^n$.

Following the ideas of Nakamura, Reid introduced in [Rei97] the scheme G-Hilb , the fine moduli space of all G-clusters. It comes equipped with a Hilbert-Chow morphism $G\text{-Hilb } \mathbb{C}^n \to X$ which sends each G-cluster to its set-theoretic support. The main irreducible component of $G\text{-Hilb } \mathbb{C}^n$ birational to X can be identified (e.g. [IN00], §2) with the scheme $\text{Hilb}^{\,G} \mathbb{C}^n$ introduced by Nakamura and Ito in [IN96]. They then proceeded to show that for G a finite subgroup of $\text{SL}_2(\mathbb{C})$, the scheme $\text{Hilb}^{\,G} \mathbb{C}^n$ is the unique crepant minimal resolution of $\mathbb{C}^2/G$.

Then Nakamura showed by explicit toric geometry computations [Nak00] that for G a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$, the scheme $\mathrm{Hilb}^{\,G}\,\mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$. He conjectured that the same is true for the non-abelian case.

This conjecture was settled by Bridgeland, King and Reid in [BKR01]. They use derived category methods and establish a category equivalence $D(Y) \to D^G(\mathbb{C}^n)$ between the bounded derived categories of coherent sheaves on $Y = \mathrm{Hilb}^{\,G}\,\mathbb{C}^n$ and of G-equivariant coherent sheaves on $\mathbb{C}^n$, respectively. Under a certain assumption on the dimension of the fibers of Y, which holds automatically when $n \leq 3$, they prove that the Fourier-Mukai transform which uses the structure sheaf of the universal G-cluster $\mathcal{U}_G \subset Y \times \mathbb{C}^n$ is the requisite equivalence. In particular, this shows that Y is a crepant resolution of X, proving Nakamura's conjecture. It is then further shown ([BKR01], §8) that in the case of $n = 3$, $\mathrm{Hilb}^{\,G}\,\mathbb{C}^3$ is the only component of G-Hilb $\mathbb{C}^3$, i.e. G-Hilb $\mathbb{C}^3$ is connected. In dimension two this was proven by Ishii in [Ish02], while in dimensions four and higher it is known to be false.

For $n \geq 3$ crepant resolutions of $\mathbb{C}^n/G$, if they exist, are not necessarily unique. The question arose whether G-clusters can be generalised further, to obtain the other crepant resolutions by a moduli space construction. Subsequent research had shown that it was not necessary to give an orbit a subscheme structure - it is sufficient to equip an orbit with a coherent sheaf that looks like what we would expect of an image of a skyscraper sheaf of a point under a derived category equivalence as above. This generalisation was a concept of a G-constellation given by Craw in his thesis [Cra01]:

Definition 1.2. A G-`constellation` is a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$, whose global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$ form a regular representation of G.

Note that a priori a definition of G-constellation doesn't exclude sheaves supported at more than one orbit of G. However a *gnat*-family consists only of those supported at a single orbit.

Observe that, tautologically, the structure sheaf of any G-cluster is a G-constellation. In fact on a free orbit this all we get: the concepts of a G-constellation, a G-cluster and a set-theoretic orbit coincide where G acts freely. At the origin, however, there are many G-constellations which do not arise as structure sheaves of G-clusters. Too many in fact: the moduli space of all G-constellations is non-separated at the origin, suggesting that some sort of stability conditions are needed.

These came to us courtesy of a natural 1-to-1 correspondence existing between G-constellations and representations of the McKay quiver of G into the regular representation of G. This allows for the use of an earlier result of King [Kin94] on GIT construction of moduli spaces of quiver representations to introduce the stability conditions known as θ-stability on G-constellations and to construct for any given stability condition θ a moduli space M_θ of θ-stable G-constellations together with a projective morphism to X and a universal θ-stable G-constellation $\mathcal{U}_\theta$ in $\mathbf{Coh}\, Y \times \mathbb{C}^n$. In a quiver-theoretic context, Kronheimer [Kro89] had already considered these moduli spaces and have studied the chamber structure in the space Π of stability parameters θ, where all values of θ in the same chamber yield the same M_θ. The methods of [BKR01] can be then extended to show that, under the same assumptions on the fiber dimensions of M_θ, the Fourier-Mukai transform $D(M_\theta) \to D^G(\mathbb{C}^n)$ is an equivalence of categories, which makes the main irreducible component of M_θ a crepant resolution of $\mathbb{C}^n/G$. In case of an abelian G, an explicit description of this coherent component is provided in toric terms by Craw, Maclagan and Thomas in [CMT05a], [CMT05b].

Craw in his thesis conjectured that when G is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ every crepant resolution projective over $\mathbb{C}^3/G$ can be realised as a moduli space M_θ of θ-stable G-constellations for some chamber in Π. In the case of G being abelian, this was proved by Craw and Ishii in [CI04].

Thus one motivation for the study of families of G-constellations on a fixed resolution Y is an observation that, as evident from [CI04], there exist stability parameters θ for which the GIT construction yields isomorphic moduli spaces M_θ, but equips them with different tautological families of G-constellations $\mathcal{U}_\theta$. Another is the desire to obtain for a given crepant resolution Y a direct construction of the derived McKay equivalence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ as a Fourier-Mukai functor using an appropriate G-constellation family. Finally, the question of a moduli construction of non-projective (over X) crepant resolutions still remains open.

2 Gnat-Families

Rather than constructing a resolution as a moduli space of G-constellations, we take an arbitrary (not necessarily projective or crepant) resolution of X and study the flat families of G-constellations that it can parametrise.

We would like for a family of G-constellations to be a flat $\mathcal{O}_Y$-module, whose restriction to any point of Y would give us the respective G-constellation. From this point of view, it would be better to consider, instead of the whole G-constellation $\mathcal{F}$, just its space of global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$. It is a vector space V with G and R actions, satisfying

$$g.(f.\mathbf{v}) = (g.f).(g.\mathbf{v}) \tag{2.1}$$

As $\mathbb{C}^n$ is affine, functor $(\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ recovers $\mathcal{F}$ from $\Gamma(\mathbb{C}^n, \mathcal{F})$, and (2.1) defines the G-equivariant structure.

It is convinient to view such vector spaces as modules for the following non-commutative algebra:

Definition 2.1. A cross-product algebra $\mathrm{R} \rtimes \mathrm{G}$ is an algebra, which has the vector space structure of $R \otimes_{\mathbb{C}} \mathbb{C}[G]$ and the product defined by setting, for all $g_1, g_2 \in G$ and $f_1, f_2 \in R$,

$$(f_1 \otimes g_1) \times (f_2 \otimes g_2) = (f_1(g_1.f_2)) \otimes (g_1 g_2) \tag{2.2}$$

Functors $\tilde{\bullet} = (\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ and $\Gamma(\mathbb{C}^n, \bullet)$ give an equivalence between the categories of R $\rtimes$ G-modules and of quasi-coherent G-equivariant sheaves on $\mathbb{C}^n$.

This is not a pure formalism - R $\rtimes$ G is one of the *non-commutative crepant resolutions* of $\mathbb{C}^n/G$, a certain class of non-commutative algebras introduced by Michel van den Bergh in [dB02] as an analogue of a commutative crepant resolution for an arbitrary non-quotient Gorenstein singularity. For three-dimensional terminal singularities, van den Bergh shows ([dB02], Theorem 6.3.1) that if a non-commutative crepant resolution Q exists, then it is possible to construct commutative crepant resolutions as moduli spaces of certain stable Q-modules.

Under $\Gamma(\mathbb{C}^n, \bullet)$, to G-constellations correspond R $\rtimes$ G-modules, which are isomorphic, as representations of G, to the regular representation V_{reg}. By abuse of notation, we shall use the term G-constellations to also mean such R $\rtimes$ G-modules. This interpretation allows us to define a family of G-constellations as a locally-free sheaf on Y, instead of $Y \times \mathbb{C}^n$:

Definition 2.2. `A family of` G`-constellations parametrised by` Y is a sheaf $\mathcal{F}$ of $(\mathrm{R} \rtimes \mathrm{G}) \otimes_{\mathbb{C}} \mathcal{O}_Y$-modules on Y, locally free as an $\mathcal{O}_Y$-module, such that, for any point $\iota_p : p \to Y$, the fiber $\mathcal{F}_{|p} = \iota_p^* \mathcal{F}$ is a G-constellation.

We wish to develop a notion of a geometrically natural family, in which for any $p \in Y$ the G-constellation $\mathcal{F}_{|p}$ would be geometrically related to the G-orbit $q^{-1}\pi(p)$. For example, the G-constellation $\tilde{\mathcal{F}_{|p}}$, as a sheaf on $\mathbb{C}^n$, is supported on a finite union of G-orbits. We could ask, mimicking the moduli spaces M_θ of θ-stable G-constellations and their tautological families, for this support to be precisely $q^{-1}\pi(p)$.

This turns out to be enough to warranty a much wider range of naturality properties.

Definition 2.3. A `generically natural` family of G-constellations parametrised by Y (or a `gnat-family`, for short) is a family $\mathcal{F}$ of G-constellations, such that for every $p \in Y$

$$\mathrm{Supp}_{\mathbb{C}^n}(\mathcal{F}_{|p}) = q^{-1}\pi(p)$$

Proposition 2.4. *Let $\mathcal{F}$ be a family of G-constellations parametrised by Y. Then the following are equivalent:*

1. *On any $U \subset Y$, such that πU consists of free orbits, $\mathcal{F}$ is equivalent (locally isomorphic) to $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$.*

2. *There exists a* $(\mathrm{R} \rtimes \mathrm{G}) \otimes_{\mathbb{C}} K(Y)$*-module isomorphism:*

$$\mathcal{F}_{|p_Y} \xrightarrow{\sim} (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{p_Y}$$

where p_Y *is the generic point of* Y*.*

3. *There exists an* $(\mathrm{R} \rtimes \mathrm{G}) \otimes_{\mathbb{C}} \mathcal{O}_Y$*-module embedding*

$$F \hookrightarrow K(\mathbb{C}^n)$$

where $\mathcal{O}_Y$*-module structure on* $K(\mathbb{C}^n)$ *is induced by the map* $q : Y \to X$*.*

4. $\mathcal{F}$ *is a gnat-family.*

5. *The action of* $(\mathrm{R} \rtimes \mathrm{G}) \otimes_{\mathbb{C}} \mathcal{O}_Y$ *on* $\mathcal{F}$ *descends to the action of* $(\mathrm{R} \rtimes \mathrm{G}) \otimes_{R^G} \mathcal{O}_Y$*, where the* R^G*-module structure on* $\mathcal{O}_Y$ *is induced by the map* $q : Y \to X$*.*

Sketch. Implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ are quite straightforward. The interesting one is $5 \Rightarrow 1$.

Consider a natural algebra homomorphism

$$\Psi : \ (\mathrm{R} \rtimes \mathrm{G}) \otimes_{R^G} \ \mathcal{O}_Y \to \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$$

LHS is isomorphic to $\pi^* \, \mathcal{E}nd_{\mathcal{O}_X}(q_* \mathcal{O}_{\mathbb{C}^n})$. Over U, as q is flat over πU, LHS is further isomorphic to $\mathcal{E}nd_{\mathcal{O}_Y}(\pi^* q_* \mathcal{O}_{\mathbb{C}^n})$. Thus we have

$$\Psi' : \mathcal{E}nd_{\mathcal{O}_U}(\pi^* q_* \mathcal{O}_{\mathbb{C}^n}) \to \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{F})$$

It is a homomorphism of (split) Azumaya algebras of the same constant rank, which is an isomorphism on the centers. Hence Ψ' must be an isomorphism itself. Then, by Skolem-Noether theorem, Ψ' must locally be induced by isomorphisms $\pi^* q_* \mathcal{O}_{\mathbb{C}^n} \to \mathcal{F}$.

□

3 G-divisors

Since G is abelian, any family $\mathcal{F}$ of G-constellations on Y splits into invertible eigensheaves: $\mathcal{F} = \oplus_{\chi \in G^\vee} \mathcal{F}_\chi$. If $\mathcal{F}$ is also a gnat-family, then it can be embedded into $K(\mathbb{C}^n)$. Now, generally, on a scheme S an invertible sheaf embedded into $K(S)$ defines a Cartier divisor on S.

Therefore, just as the group $K^*_G(\mathbb{C}^n)^*$ of the invertible G-homogeneous elements of $K(\mathbb{C}^n)$ extends $K^*(Y)$:

$$1 \to K^*(Y) \to K^*_G(\mathbb{C}^n) \xrightarrow{\rho} G^\vee \to 1 \tag{3.1}$$

we extend the group of Cartier divisors on Y as follows:

Definition 3.1. A rational function $f \in K^*(\mathbb{C}^n)$ is said to be G-homogeneous (of weight χ), if there exists a character $\chi \in G^\vee$ such that

$$g.f = \chi(g)f \qquad \forall\, g \in G$$

Definition 3.2. A G-`Cartier divisor` on Y is a global section of the sheaf of multiplicative groups $K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$, where the sheaf $K_G^*(\mathbb{C}^n)$ is the constant sheaf on Y of the G-homogeneous elements of $K(\mathbb{C}^n)$ and the sheaf $\mathcal{O}_Y^*$ is the sheaf of invertible regular functions on Y.

Similar to the ordinary Cartier divisors, a G-Cartier divisor can be specifed by a set of pairs (U_i, f_i), where U_i are an open cover of Y and f_i are G-homogenous rational functions on $\mathbb{C}^n$, such that for any i and j, f_i/f_j defines an invertible regular function on $U_i \cap U_j$.

As with ordinary Cartier divisors, we say that a G-Cartier divisor is principal if it lies in the image of the natural map $K_G^*(\mathbb{C}^n) \to K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$ and call two divisors linearly equivalent if their difference is principal.

Thus, we obtain a short exact sequence of abelian groups:

$$1 \to \mathrm{Car}(Y) \to G\text{-}\mathrm{Car}(Y) \xrightarrow{\rho} G^\vee \to 1 \tag{3.2}$$

We call an image of a Cartier divisor D under the map ρ its `weight` and say that D is a $\rho(D)$-Cartier divisor.

The construction of the invertible subsheaf $\mathcal{L}(D)$ of $K(Y)$ corresponding to a Cartier divisor D, extends naturally to a construction of an invertible subsheaf $\mathcal{L}(D)$ of $K_G^*(\mathbb{C}^n)$ corresponding to a G-Cartier divisor D.

Proposition 3.3. *The map $D \to \mathcal{L}(D)$ gives an isomorphism between G-$\mathrm{Car}\,Y$ and the group of invertible G-subsheaves of $K(\mathbb{C}^n)$. Furthermore, it descends to an isomorphism of the group G-Cl of G-Cartier divisors up to linear equivalence and the group G-Pic of invertible G-sheaves on Y.*

We now seek to define a matching notion of a G-Weil divisor. The key notion is: valuations at prime divisors of Y define a unique group homomorphism val_K from $K^*(Y)$ to $\mathrm{Div}\,Y$, the group of Weil divisors. Looking at the short exact sequence (3.1), we see that val_K must extend uniquely to a homomorphism val_{K_G} from $K_G^*(\mathbb{C}^n)$ to $\mathbb{Q}$-$\mathrm{Div}\,Y$, as $G^\vee$ is finite and $\mathbb{Q}$ is injective. We further obtain a quotient homomorphism $val_{G^\vee}$ from $G^\vee$ to $\mathbb{Q}/\mathbb{Z}$-$\mathrm{Div}\,Y$.

The short exact sequence (3.2) now becomes a commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Car}\,Y & \longrightarrow & G\text{-}\mathrm{Car}\,Y & \xrightarrow{\rho} & G^\vee & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle val_K} & & \downarrow{\scriptstyle val_{K_G}} & & \downarrow{\scriptstyle val_{G^\vee}} & & \\
0 & \longrightarrow & \mathrm{Div}\,Y & \longrightarrow & \mathbb{Q}\text{-}\mathrm{Div}\,Y & \longrightarrow & \mathbb{Q}/\mathbb{Z}\text{-}\mathrm{Div}\,Y & \longrightarrow & 0
\end{array} \tag{3.3}$$

Aiming to have a short exact sequence similar to (3.2), we now define the group G-Div Y of G-Weil divisors to be the subgroup of $\mathbb{Q}$-Div Y, which consists of the pre-images of $val_{G^\vee}(G^\vee) \subset \mathbb{Q}/\mathbb{Z}$-Div Y.

We call a G-Weil divisor principal if it is an image of a single function $f \in K_G^*(\mathbb{C}^n)$ under val_{K^G}, call two G-Weil divisors linearly equivalent if their difference is principal and call a divisor $\sum q_i D_i$ effective if all $q_i \geq 0$.

We now have a following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & \operatorname{Car} Y \arrow[r] \arrow[d, "val_K"'] & G\text{-}\operatorname{Car} Y \arrow[r, "\rho"] \arrow[d, "val_{K_G}"'] & G^\vee \arrow[r] \arrow[d, "val_{G^\vee}"'] & 1 \\ 0 \arrow[r] & \operatorname{Div} Y \arrow[r] & G\text{-}\operatorname{Div} Y \arrow[r] & val_{G^\vee}(G^\vee) \arrow[r] & 0 \end{tikzcd} \tag{3.4}$$

A priori there is no reason for val_{K_G} in (3.4) to be an isomorphism. Indeed, although all the definitions above make sense for a general scheme Y birational to X, simply assuming Y to be smooth is not enough to warranty G-Cartier and G-Weil divisors to be isomorphic or even well-behaved. For an example let Y be the smooth locus of X. It can be shown, that while val_K is an isomorphism, val_{K_G} is not even injective as G-Car Y has torsion. And $val_{G^\vee}$ is the zero map, thus G-Div Y is just Div Y.

Proposition 3.4. *If Y is smooth and proper over X, then val_K, val_{K_G} and $val_{G^\vee}$ in (3.4) are all isomorphisms.*

4 Classification of the *gnat*-Families

Given a *gnat*-family $\mathcal{F} = \oplus \mathcal{F}_\chi$, we can embed it into $K(\mathbb{C}^n)$. An image of $\mathcal{F}_\chi$ under such an embedding is an invertible subsheaf of $K_G^*(\mathbb{C}^n)$ and therefore the embedding defines a unique G-Weil divisor set $\{D_\chi\}_{\chi \in G^\vee}$ on Y such that the image of $\mathcal{F}$ in $K(\mathbb{C}^n)$ is $\oplus \mathcal{L}(-D_\chi)$.

Conversely, given a G-divisor set $\{D_\chi\}_{\chi \in G^\vee}$ such that each D_χ is a χ-Weil divisor, we could ask when is $\oplus \mathcal{L}(-D_\chi)$ a *gnat*-family.

Proposition 4.1. *Let $\{D_\chi\}_{\chi \in G^\vee}$ be as above. Then $\oplus \mathcal{L}(-D_\chi)$ is a gnat-family if and only if for any G-homogeneous $f \in R$ and any $\chi \in G^\vee$ we have*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \tag{4.1}$$

where $\rho(f) \in G^\vee$ is the weight of f.

NB: Observe that the condition (4.1) is equivalent to a set of $|G|$ inequalities for each prime Weil divisor, and that these sets are all independent of each other.

We call G-divisor sets $\{D_\chi\}_{\chi \in G^\vee}$ which satisfy (4.1) the `reductor sets`.

Recall that in moduli problems it is a standard practice to consider the families up to equivalence, that is up to a local isomorphism.

Theorem 4.1. *The isomorphism classes of gnat-families on Y are in a one-to-one correspondence with the linear equivalence classes of the reductor sets $\{D_\chi\}$. The equivalence classes of gnat-families on Y are in a one-to-one correspondence with the reductor sets $\{D_\chi\}$, in which $D_{\chi_0} = 0$.*

We say that a reductor set $\{D_\chi\}$ is `normalised`, if $D_{\chi_0} = 0$.

Proposition 4.2 (Canonical family). *Define the divisor set $\{D_\chi\}$ by*

$$D_\chi = \sum_P v(P, \chi) P$$

Then $\{D_\chi\}$ is a normalised reductor set. Moreover, the corresponding family $\oplus \mathcal{L}(-D_\chi)$ is the pushdown to Y of the structure sheaf of the normalization of the reduced fibre product $Y \times_X \mathbb{C}^n$.

Proposition 4.3 (Maximal shift family). *Define the divisor set $\{M_\chi\}$ by*

$$M_\chi = \sum_P \min_{f \in R_\chi} v_P(f)\ P$$

Then $\{M_\chi\}$ is a normalised reductor set.

NB: It can be shown that, for any $\chi \in G^\vee$, the coefficient of M_χ at a prime Weil divisor P is non-zero if and only if P is exceptional or the image of P in X is the branch divisor of the quotient map $\mathbb{C}^n \to X$. Therefore, for each $\chi \in G^\vee$, the coefficient of M_χ is non-zero at only finitely many prime divisors in Y.

Proposition 4.4. *Let $\{D_\chi\}$ be any normalised reductor set. Then*

$$-M_{\chi^{-1}} \leq D_\chi \leq M_\chi$$

for any $\chi \in G^\vee$.

Corollary 4.5. *The number of equivalence classes of gnat-families is finite.*

We summarise our results in the following theorem:

Theorem 4.2 (Classification of *gnat*-families). *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G, Y nonsingular and $\pi : Y \to X$ a proper birational map. Then isomorphism classes of gnat-families on Y are in 1-to-1 correspondence with linear equivalence classes of G-divisor sets $\{D_\chi\}_{\chi \in G^\vee}$, each D_χ a χ-Weil divisor, which satisfy the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall\, \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Such a divisor set $\{D_\chi\}$ corresponds then to a gnat-family $\bigoplus \mathcal{L}(-D_\chi)$.

This correspondence descends to a 1*-to-*1 *correspondence between equivalence classes of gnat-families and sets* $\{D_\chi\}$ *as above and with* $D_{\chi_0} = 0$. *Furthermore, each divisor* D_χ *in such a set satisfies inequality*

$$-M_{\chi^{-1}} \leq D_\chi \leq M_\chi$$

where $\{M_\chi\}$ *is a fixed divisor set defined by*

$$M_\chi = \sum_P (\min_{f \in R_\chi} v_P(f))P$$

As a consequence, the number of equivalence classes of gnat-families is finite.

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