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Projective manifolds with hyperplane sections being five-sheeted covers of projective space*

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ABSTRACT

Here we consider the classification problem of smooth complex projective varieties containing a finite branched covering of $\mathbb{P}^n$ as a very ample divisor. In this talk, we first introduce the problem and its background. After that, we present a classification result (Main Theorem) in case where the degree of the finite covering is five, and mention the keys of the proof.

1 Introduction and Main Theorem

(**1.1**) Let X be a smooth complex projective $(n+1)$-fold and L a very ample line bundle on it. We consider the following condition:

$(*)_d$ There exists a smooth member $A \in |L|$ such that there exists a finite morphism $\pi\colon A \to \mathbb{P}^n$ of degree d.

(**1.2**) We immediately find that the following are examples of (X, L):

Examples (Obvious pairs). $(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$ and $(H_d^{n+1}, \mathcal{O}_{H_d^{n+1}}(1))$, where $H_d^{n+1} \subset \mathbb{P}^{n+2}$ is a smooth hypersurface of degree d, and π is a projection from a point.

So, what kind of "non-obvious" pairs show up? This natural question underlies our study.

(**1.3**) Classical results on surfaces with hyperelliptic curves as hyperplane sections (e.g. [Cas]) and their revision made in the eighties ([Ser], [SV]) called the attention to the problem of classifying pairs (X, L) with $(*)_d$. The problem has been considered by several authors:

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- In the case of $(n, d) = (1, 2)$, F. Serrano [Ser], A. J. Sommese-A. Van de Ven [SV] classified the pairs (X, L) with $(*)_d$ independently.

- In the case of $(n, d) = (1, 3)$, M. L. Fania [Fan] studied the structure of the pairs (X, L) by using the adjunction mapping.

- In cases where (i) $(n \geq 2, d = 2)$ and (ii) $(n \geq 4, d = 3)$, the pairs (X, L) are classified by A. Lanteri-M. Palleschi-A. J. Sommese (L-P-S for short) in [LPS1] and [LPS2].

(**1.4**) We want to classify the polarized pairs (X, L) for all n and d. But it seems to be a difficult problem.

In what follows, we assume that $n > d$. Then we have the following diagram:

$$\begin{array}{ccccccc} \mathrm{Pic}(X) & \xrightarrow{i^*} & \mathrm{Pic}(A) & \xleftarrow{\pi^*} & \mathrm{Pic}(\mathbb{P}^n) & \cong & \mathbb{Z} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \\ \mathcal{H} & \longleftrightarrow & \pi^*\mathcal{O}_{\mathbb{P}^n}(1) & \longleftrightarrow & \mathcal{O}_{\mathbb{P}^n}(1), & & \end{array}$$

where $i\colon A \hookrightarrow X$ denotes the inclusion map. Here we obtain that

- π^* is an isomorphism by virtue of a Barth-type theorem by R. Lazarsfeld ([Laz, Proposition 3.1]); and

- i^* is an also isomorphism by the Lefschetz hyperplane section theorem.

Thus we see that there exists a unique line bundle $\mathcal{H} \in \mathrm{Pic}(X)$ such that $\mathcal{H}_A = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. And one can easily show that each of the above three line bundles is an ample generator of each of the above Picard groups.

(**1.5**) For $d \in \{2, 3\}$, the pairs (X, L) with $(*)_d$ are classified by L-P-S as mentioned in (**1.3**). The result on the degree $d = 2$ case says that there are no pairs (X, L) except the obvious ones. And, in the degree 3 case, the "non-obvious" pair is only one:

Theorem 1 (Lanteri-Palleschi-Sommese). *Let X be a projective manifold of dimension $n+1$ and L a very ample line bundle on X satisfying $(*)_d$.*

(1) [LPS1] *If* $n > d = 2$, *then* (X, L) *is one of the following*:
(i) $(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(2))$; *or* (ii) $(\mathbb{Q}^{n+1}, \mathcal{O}_{\mathbb{Q}^{n+1}}(1))$.

(2) [LPS2] *If* $n > d = 3$, *then* (X, L) *is one of the following*:
(i) $(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(3))$; (ii) $(H_3^{n+1}, \mathcal{O}_{H_3^{n+1}}(1))$; *or* (iii) $(Y, 3\mathcal{L})$, *where* $(Y, \mathcal{L})$ *is a Del Pezzo manifold of degree one.*

Note. (i) **Definition.** For a polarized manifold $(Y, \mathcal{L})$ of $\dim Y = n+1$,

$$(Y, \mathcal{L}): \text{ a Del Pezzo manifold of degree one}$$
$$\overset{\text{def}}{\Longleftrightarrow} -K_Y = n\mathcal{L} \text{ and the degree } d(Y, \mathcal{L}) := \mathcal{L}^{n+1} = 1.$$

(ii) In (2), the very ampleness of $3\mathcal{L}$ is also proved by L-P-S.

(**1.6**) It is natural to ask what kind of "non-obvious" pairs appear as the degree of the covering π is getting large. So we obtained a classification of the pairs (X, L) with $(*)_d$ in the degree 5 case. In fact, no less than two "non-obvious" pairs newly show up as stated below: (4) and (5).

Main Theorem. *Let X be a projective manifold of dimension $n+1 > 6$. Then there exists a very ample line bundle L on X that satisfies the condition $(*)_{d=5}$ if and only if (X, L) is one of the following*:

(1) $(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(5))$;

(2) $(H_5^{n+1}, \mathcal{O}_{H_5^{n+1}}(1))$;

(3) $(Y, 5\mathcal{L})$;

(4) $(V_{10}, \mathcal{O}_{V_{10}}(5))$, *where V_{10} is a smooth weighted hypersurface of degree* 10 *in the weighted projective space* $\mathbb{P}(5, 2, 1^{n+1})$; *or*

(5) $(W_{20}, \mathcal{O}_{W_{20}}(5))$, *where W_{20} is a smooth weighted hypersurface of degree* 20 *in* $\mathbb{P}(5, 4, 1^{n+1})$.

Notation. Here $\mathbb{P}(e_0, \ldots, e_N)$ denotes a weighted projective space $\operatorname{Proj} \mathbb{C}[X_0, \ldots, X_N]$ with weights $\mathrm{wt}(X_i) = e_i$ for each $0 \le i \le N$. And, for instance, we abbreviate $\mathbb{P}(5, 2, \underbrace{1, \ldots, 1}_{m})$ to $\mathbb{P}(5, 2, 1^m)$.

(**1.7**) A new problem arising in our degree 5 case is to determine the structure of a certain polarized manifold $(X, \mathcal{H})$ with invariants

$$\Delta(X,\mathcal{H}) = d(X,\mathcal{H}) = 1,$$

where $\Delta(X,\mathcal{H})$, said to be the *Δ-genus* of $(X,\mathcal{H})$, is defined by $\dim X + \mathcal{H}^{\dim X} - h^0(X,\mathcal{H})$.

In general, the polarized manifolds with these invariants are classified by T. Fujita [Fuj1] in case where the sectional genus

$$g(X,\mathcal{H}) := 1 + \frac{1}{2}(K_X + n\mathcal{H}) \cdot \mathcal{H}^n \leq 2.$$

In case where the sectional genus ≥ 3, the classification problem of the polarized manifolds with $\Delta(X,\mathcal{H}) = d(X,\mathcal{H}) = 1$ is yet to be studied and developed for no less than about two decades.

In the degree 3 case, it follows from the arguments as in [LPS2] that $g(X,\mathcal{H}) = 1$. Therefore one has only to apply the Fujita's classification in [Fuj1] in order to obtain the classification described above.

In contrast, in our degree 5 case, we have to treat the case where $g(X,\mathcal{H}) = 6$. In fact, we can determine the structure of a new polarized manifold which does not appear in the Fujita's classification ((5) in Main Theorem), successfully.

2 The Keys of the Proof of Main Theorem

(2.1) The following are the keys of the proof.

(A) To prove the very ampleness of the line bundle $\mathcal{O}_{W_{20}}(5)$ in Main Theorem (5) ('if' part).

(B) To determine the structure of a certain polarized manifold $(X,\mathcal{H})$ with $\Delta(X,\mathcal{H}) = d(X,\mathcal{H}) = 1$ and $g(X,\mathcal{H}) = 6$ ('only if' part).

(2.2) What is the important is the following:

Fact 1. *For a polarized manifold $(M,\mathcal{L})$,*

$$\Delta(M,\mathcal{L}) = d(M,\mathcal{L}) = 1 \Longrightarrow \mathrm{Bs}|\mathcal{L}| \text{ consists of a single point.}$$

For the proof, we refer to [Fuj1, (13.6)]. In what follows, we put $p := \mathrm{Bs}|\mathcal{L}|$.

PART (A)

(**2.3**) We take a coordinate system of $W_{20} \subset \mathbb{P}(5,4,1^{n+1})$, $\{x, y, z_0, \dots, z_n\}$, where $\mathrm{wt}(x, y, z_i) = (5, 4, 1)$ for each $0 \le i \le n$. Then it is shown that the following form a basis of $H^0(W_{20}, \mathcal{O}_{W_{20}}(5))$:

$$x, yz_0, \dots, yz_n, z_{j_1} \cdots z_{j_5}, \text{ where } 0 \le j_1 \le \cdots \le j_5 \le n.$$

(**2.4**) We get the conclusion with the next steps:

(A1) We show that $\mathrm{Bs}|\mathcal{O}_{W_{20}}(5)| = \emptyset$;

(A2) We claim that the morphism $\varphi\colon W_{20} \to \mathbb{P}(|\mathcal{O}_{W_{20}}(5)|)$ associated to $\mathcal{O}_{W_{20}}(5)$ is injective; and

(A3) We prove that the linear system $|\mathcal{O}_{W_{20}}(5)|$ separates the tangent vectors.

(**2.5**) We use the following fact:

Fact 2 (A. Laface). *Let $(M, \mathfrak{L})$ be a polarized manifold with $\mathfrak{L} \ge 0$. Assume that the graded ring $R(M, \mathfrak{L}) := \bigoplus_{l=0}^{\infty} H^0(M, l\mathfrak{L})$ is generated in degrees $\le r$. Then the rational map associated to $r\mathfrak{L}$*

$$\varphi_{r\mathfrak{L}}\colon M \setminus \mathrm{Bs}|\mathfrak{L}| \longrightarrow \mathbb{P}(|r\mathfrak{L}|)$$

gives an embedding.

For the proof, see [Laf, Theorem 2.2].

From $\Delta(W_{20}, \mathcal{O}_{W_{20}}(1)) = d(W_{20}, \mathcal{O}_{W_{20}}(1)) = 1$, it suffices to prove that φ gives an embedding at p.

(**2.6**) (A1): Since W_{20} does not meet the singular locus of $\mathbb{P}(5,4,1^{n+1})$, we have

$$W_{20} \supset \mathrm{Bs}|\mathcal{O}_{W_{20}}(5)| = (x = 0) \cap \Big(\bigcap_{0 \le i \le n} (z_i = 0)\Big) = \emptyset.$$

(**2.7**) (A2): For any $q \in W_{20}$ satisfying $\varphi(p) = \varphi(q)$, we obtain that $z_i(q) = 0$ for all $0 \le i \le n$. Hence, by (2.2), we see that

$$q \in \bigcap_{0 \le i \le n} (z_i = 0) = \mathrm{Bs}|\mathcal{O}_{W_{20}}(1)| = \{p\}.$$

(**2.8**) (A3): To prove is that for any non-zero tangent vector $\tau \in T_p(W_{20})$ there exists a section $\sigma \in H^0(\mathcal{O}_{W_{20}}(5))$ satisfying the following conditions

$$\sigma(p) = 0 \text{ and } d\sigma(\tau) \neq 0.$$

We claim that $\sigma_i := yz_i$ satisfies the above conditions for some $0 \leq i \leq n$. The former condition holds because $z_i(p) = 0$ for all $0 \leq i \leq n$. We prove that the latter holds by contradiction. Suppose that there exists an non-zero $\tau \in T_p(W_{20})$ such that $d\sigma_i(\tau) = 0$ for all i. Since $d\sigma_i(\tau) = y(p)dz_i(\tau)$ and $y(p) \neq 0$, we see that $dz_i(\tau) = 0$ for each i. Therefore we obtain

$$\tau \in T_p(\Gamma), \text{ where } \Gamma := \bigcap_{1 \leq i \leq n} (z_i = 0) \subset W_{20}.$$

Furthermore, since $dz_0(\tau) = 0$, we obtain that $\Gamma \cdot \mathcal{O}_{W_{20}}(1) \geq 2$, which contradicts $\mathcal{O}_{W_{20}}(1)^{n+1} = 1$. Thus (A3) is proved.

Part (B)

(**2.9**) The aim of Part (B) is to determine the structure of the polarized manifold $(X, \mathcal{H})$ with the following condition:

$$h^0(A, \mathcal{H}_A) = n + 1 \text{ and } g(X, \mathcal{H}) = 6.$$

Here we want to prove the following:

Theorem 2. *We have*

$$(X, \mathcal{H}) \cong (W_{20}, \mathcal{O}_{W_{20}}(1)).$$

(**2.10**) The next lemma follows from typical calculations. What is the important in the proof of Theorem 2 is the very ampleness of $5\mathcal{H}$.

Lemma 1. *We have* (1) $L = 5\mathcal{H}$; (2) $d(X, \mathcal{H}) = 1$; *and* (3) $h^0(X, \mathcal{H}) = n + 1$, *therefore* $\Delta(X, \mathcal{H}) = 1$.

Remark. Let V_i $(1 \leq i \leq n)$ be a general member of $|\mathcal{H}|$. Set

$$S := \bigcap_{1 \leq i \leq n-1} V_i \subset X, \text{ and put } C := S \cap V_n \subset X.$$

Then, in fact, we can show that S and C are smooth. Moreover, we obtain that $h^0(\mathcal{H}_S) = 2$ and $h^0(\mathcal{H}_C) = 1$. But we omit the proof here.

(**2.11**) We proceed with the following steps.

(B1) We show that $R(C, \mathcal{H}_C) \cong \mathbb{C}[x, y, z]/(F_{20})$, where

- $\mathrm{wt}(x, y, z) = (5, 4, 1)$; and
- F_{20} is some irreducible homogeneous polynomial of degree 20 in $\mathbb{C}[x, y, z]$.

(B2) We claim that the restriction map $\rho\colon R(S, \mathcal{H}_S) \to R(C, \mathcal{H}_C)$ is surjective.

Indeed, (B1) and (B2) imply the assertion by the following reason: We see that $(S, \mathcal{H}_S)$ is a weighted hypersurface of degree 20 in $\mathbb{P}(5, 4, 1^2)$. Combining this and a result by S. Mori [Mor, Proposition 3.10], we obtain that $(X, \mathcal{H})$ is a weighted hypersurface of degree 20 in $\mathbb{P}(5, 4, 1^{n+1})$. Thus the assertion holds.

From now on, we explain the idea for the proof of (B1) and (B2).

(**2.12**) *Outline of the proof of* (B1): First we will find generators of the graded algebra $R(C, \mathcal{H}_C)$ by calculating each $h^0(l\mathcal{H}_C)$. Using the adjunction formula, we obtain that $g(C) = 6$. And it follows from the Riemann-Roch theorem for curves that

$$h^0(l\mathcal{H}_C) - h^0((10 - l)\mathcal{H}_C) = l - 5 \quad \text{for all } l \geq 0.$$

For $l \geq 11$, we immediately have $h^0(l\mathcal{H}_C) = l - 5$ due to the Kodaira vanishing theorem. For $0 \leq l \leq 10$, one can compute the values $h^0(l\mathcal{H}_C)$ as follows:

l	$h^0(l\mathcal{H}_C)$	l	$h^0(l\mathcal{H}_C)$
1	1	6	3
2	1	7	3
3	1	8	4
4	2	9	5
5	3	10	6

In fact, the keys of the computation of $h^0(l\mathcal{H}_C)$ are $d(X, \mathcal{H}) = 1$ and the very ampleness of $L = 5\mathcal{H}$. We see that the image of C by the morphism associated to $5\mathcal{H}_C$ is a smooth plane quintic curve. Therefore we have $h^0(5\mathcal{H}_C) = 3$. Furthermore, it is well-known that a smooth plane quintic curve has neither g^1_2 nor g^1_3. Thus it follows that $h^0(2\mathcal{H}_C) = h^0(3\mathcal{H}_C) = 1$.

We also see that $h^0(4\mathcal{H}_C) = 2$ by using $\deg \mathcal{H}_C = 1$. In this way, we obtain the above table.

Now let z be a basis of $H^0(\mathcal{H}_C)$. Choose $y \in H^0(4\mathcal{H}_C)$ such that $H^0(4\mathcal{H}_C) = \langle z^4, y\rangle$. Moreover, choose $x \in H^0(5\mathcal{H}_C)$ such that $H^0(5\mathcal{H}_C) = \langle z^5, yz, x\rangle$. It is proved that the sections x, y, z are generators of $R(C, \mathcal{H}_C)$ by using the following fact:

Fact 3. *For any integer $l \geq 12$, the equation $5i + 4j = l$ has at least one solution (i, j) of non-negative integers.*

Next, we find the relations among the generators. We see that there exists a unique relation F_{20} of degree 20 comparing the values of each $h^0(l\mathcal{H}_C)$ and the numbers of pieces of the monomials in each $H^0(l\mathcal{H}_C)$. Moreover, we see that the relation is written as

$$F_{20} = x^4 + y^5 + z\phi_{19}(x, y, z)$$

after we replace x and y by suitable scalar multiples, where $\phi_{19}(x, y, z)$ is a homogeneous polynomial in x, y, z of degree 19. And we can also show that F_{20} is irreducible in $\mathbb{C}[x, y, z]$ by easy calculations. Taking account of this and $\dim \mathbb{C}[x, y, z] = 3$, we see that the relations are generated by F_{20}, which proves (B1).

(**2.13**) **Notation.** Let $\mathbf{s} = \{s_0, \ldots, s_N\}$ be a minimal set of generators of $R(S, \mathcal{H}_S)$. And write

$$R(S, \mathcal{H}_S) = \mathbb{C}[s_0, \ldots, s_N]/I_{\mathbf{s}} \text{ with some homogeneous ideal } I_{\mathbf{s}}.$$

(**2.14**) *Outline of the proof of* (B2): It suffices to prove that $R(S, \mathcal{H}_S)$ is a Cohen-Macaulay ring because it is fulfilled that $H^1(l\mathcal{H}_S) = 0$ for all $l \in \mathbb{Z}$. From now on, we find a regular sequence of length 3 contained in $R(S, \mathcal{H}_S)_+ := \bigoplus_{i>0} H^0(S, l\mathcal{H}_S)$. In fact, by recalling that $h^0(\mathcal{H}_S) = 2$, we have $H^0(\mathcal{H}_S) = \langle s, t\rangle$ with $\rho(t) = z$ and $(s)_0 = C$. Then we can show that a sequence s, t is $R(S, \mathcal{H}_S)$-regular. But we omit the proof here.

Now by using the very ampleness of $L = 5\mathcal{H}$, we can prove the following.

Lemma 2. *The restriction map $\rho_5 \colon H^0(5\mathcal{H}_S) \longrightarrow H^0(5\mathcal{H}_C)$ is surjective.*

Here we choose a section $u \in H^0(5\mathcal{H}_S)$ such that $\rho_5(u) = x$. What we want to prove is that u is $R(S, \mathcal{H}_S)/(s,t)$-regular. The following is important:

Lemma 3. *The ideal I_{s} has no generators in degrees ≤ 5.*

For the proof, we refer to [Ami, Lemma 6.5].

Now, by (2.2), we find that $p = \mathrm{Proj}(R(S, \mathcal{H}_S)/(s,t))$ is an integral scheme. Hence $(R(S, \mathcal{H}_S)/(s,t))_+$ has no zero-divisors. Furthermore, due to Lemma 3, we get the conclusion.

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