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Fundamental groups and Diophantine geometry

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Consider the strategy of studying a point x on a curve X (defined over $\mathbb{Q}$) by associating to it a macroscopic geometric object. There are at least three important examples that come to mind:

(1) The Abel-Jacobi map that associates to x a line bundle $\mathcal{O}(x-b)$.

(2) The Kodaira-Parshin construction that associates to x a curve C_x (running in a family parametrized by X).

(3) The Frey-Hellegouarch construction that associates an elliptic curve to a point on a Fermat curve.

There is an obvious sense in which example (1) is much more canonical than the ad hoc methods of (2) and (3). However, (1) has never been generally applicable to finiteness theorems or the problem of determining points on hyperbolic curves.

The Abel-Jacobi map in the incarnation above is *algebraic*, as is important for Diophantine applications. Over the complex numbers, there is also an *analytic* description, whereby points of X parametrize extensions in the category of mixed Hodge structures:

$$\kappa_2^B : x \mapsto [0 \to H_1(X(\mathbb{C}), \mathbb{Z}) \to H_1(X(\mathbb{C}), \{x, b\}; \mathbb{Z}) \to \mathbb{Z} \to 0]$$

In fact, from this analytic perspective, Hain [5] constructed a lift of κ_2^B to a *higher Albanese map*:

$$\kappa^B : x \mapsto [P^B(X(\mathbb{C}); b, x)]$$

Here, $P^B(X(\mathbb{C}); b, x)$ is the set of unipotent paths from b to x:

$$P^B(X(\mathbb{C}); b, x) = \mathrm{Isom}^{\otimes}(F_b, F_x)$$

where F_x is the fiber functor

$$F_x : \mathrm{Un}(X(\mathbb{C}), \mathbb{Q}) \mapsto Vect_{\mathbb{Q}}$$

that associates to a local system $\mathcal{L}$ its stalk at x:

$$\mathcal{L} \mapsto \mathcal{L}_x.$$

Thus, the $P^B(X(\mathbb{C}); b, x)$ for varying x runs through a classifying space for torsors under the pro-unipotent fundamental group

$$U^B(X(\mathbb{C}), b) := P^B(X(\mathbb{C}); b, b)$$

where the objects live in a suitable category of *non-abelian mixed Hodge structures* [6].

This construction in the realm of Archimedean analysis has a non-Archimedean analogue over $\mathbb{Q}_p$ that gives rise to a map

$$\kappa^{dr/cr} : x \mapsto [P^{dr/cr}(X(\mathbb{Z}_p); b, x)]$$

going from $\mathbb{Z}_p$ points of X to classifying spaces for torsors under the De Rham/crystalline fundamental group

$$U^{dr/cr}(X \otimes \mathbb{Q}_p, b).$$

Globally, there is also a *topological* analogue

$$\kappa^{et/glob} : x \mapsto [P^{et/glob}(\bar{X}; b, x)]$$

that takes global points $x \in X(R)$, where $= \mathbb{Z}[1/S]$ is some ring of S-integers, to torsors under the unipotent étale fundamental group

$$U^{et/glob}(\bar{X}, b).$$

Recall that a motivic theory of such pro-unipotent fundamental groups was outlined by Deligne [1].

The eventual result of this discussion is a diagram:

$$\begin{array}{ccc} X(R) & \hookrightarrow & X(\mathbb{Z}_p) \\ \kappa^{et,glob}\big\downarrow & & \big\downarrow\kappa^{dr/cr} \\ H^1_f(\Gamma, U^{et}) & \xrightarrow{D \circ \mathrm{loc}_p} & U^{dr}/F^0 \end{array}$$

The objects in the bottom row are pro-algebraic varieties over $\mathbb{Q}_p$ that classify torsors in suitable categories. The bottom horizontal map comes from (easy) localization and (difficult) *non-abelian p-adic Hodge theory* ([3], [12]) which realizes the congruence between p-adic arithmetic topology and p-adic analysis. Eventually, we have an ambient pro-algebraic variety U^{dr}/F^0 and subsets $Im[X(\mathbb{Z}_p)]$ and $Im[H^1_f(\Gamma, U^{et}_n)]$ that have the property of containing the image of the global points in their intersection:

$$Im[X(R)] \subset Im[X(\mathbb{Z}_p)] \cap Im[H^1_f(\Gamma, U^{et})]$$

The effort to use this intersection to gain control of the global points is described in the papers [2], [8], [9], [11], and [10]. The guiding principle that emerges is that the program outlined by Grothendieck in [4] should be viewed as a *nonabelian analogue* of the conjecture of Birch and Swinnerton-dyer [7].

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