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Some topics on elliptic K3 surfaces — Profile of some beautiful K3's —

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Dedicated to Professor Jun-Ichi Igusa

1 Introduction

At the Kinosaki Symposium (Oct. 2006), we have reported our recent results on the following topics:
(i) Kummer sandwich theorem
(ii) Mordell-Weil Lattices (MWL) of some elliptic K3 with high rank.

The first topic (i) has since been published (see [22]), and so we make only a brief outline in §2, referring to the paper for the proof. For the second topic (ii), the detailed version is still in preparation. After recalling basic facts on MWL in §3, we treat a single special case:

$$E : y^2 = x^3 + t^5 + \frac{1}{t^5}$$

which has the Mordell-Weil group $E(\bar{\mathbf{Q}}(t))$ of rank 16 ($\bar{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$). In §4, we indicate how we determine the MWL together with some explicit generators $P_1, \ldots, P_{16}$, and the splitting field of this MWL.

Taking the opportunity of writing up this report in a semi-informal Proceedings of the Kinosaki Symposium, let me include a third topic:
(iii) Elliptic modular surface (EMS) of level 4 revisited.
In fact, it is this subject that has sparked my interest (more than 30 years ago) in the K3 surfaces, as well as in algebraic geometry in char $p > 0$. In §5, I record some exciting developments I have experienced around that time [—it may not be completely useless, I hope] and give an update of this old subject ([13], [14], [15]) which will contain, to my surprise(!), something new.

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The main reason why this becomes possible is due to the method of MWL, especially the height formula (see §3), established in [16] in the meantime.

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2 Kummer sandwich theorem

We fix a base field k which is algebraically closed and has $char(k) \neq 2, 3$. For the singular fibres of an elliptic surface, we follow Kodaira's notation [6], cf. [24].

Theorem 1. *Suppose X is any elliptic K3 surface with a section and with two II^*-fibres. Then there exist a unique Kummer surface $S = \mathrm{Km}(C_1 \times C_2)$ of the product of two elliptic curves C_1, C_2 and two commuting symplectic involutions $\sigma, \tau \in \mathrm{Aut}(S)$ such that (i) the quotient surface $S/\langle\sigma\rangle$ is birational to X, and (ii) the quotient surface $S/\langle\sigma, \tau\rangle$ is birational to S itself. In particular, S dominates and is dominated by X, by the rational maps of degree two:*

$$\phi : S \to X, \quad \psi : X \to S. \tag{1}$$

The motivation to study those special K3 surfaces is as follows. It is well-known that in the proof of the Torelli theorem (injectivity of the period map) by Piateckii-Shapiro-Shafarevich [10] a special role is played by the Kummer surfaces which are dense in the moduli space of polarized K3 surfaces. The elliptic K3 surfaces with two II^*-fibres are first constructed in Inose-Shioda [5] (corresponding to the above ψ) and used for proving the surjectivity of the period map in the case of singular K3 surfaces, which gives the complete classification of such surfaces. This construction is extended by Morrison [9] in a useful way. Another construction given by Inose [4] (corresponding to the above ϕ) has recently been reconsidered by Kuwata [7] (cf. [18], [21]), one reason being that it leads to elliptic K3 surfaces with high Mordell-Weil rank (see §4). Thus these K3 surfaces, very special as they may be from the moduli point of view, have various rich properties, which we think are still worth studying today. Along the same line, an approach to the notion of isogeny for K3 surfaces has been proposed in the case of singular K3 surfaces in [4], [5]. The above theorem illustrates that an elliptic K3 surface X with two II^*-fibres is "isogenous" to a Kummer surface S in a very concrete sense. By the way, it is an open question to decide whether or not the existence of a dominant rational map of K3 surfaces $X \to Y$ is symmetric with respect to X, Y as in the case of abelian varieties.

Actually Theorem 1 can be made very explicit in terms of equations. Let X be an elliptic K3 surface over the T-line $\mathbf{P}^1_T$ with two II^*-fibres. Then:

Proposition 2. *Every such X has the defining equation*

$$F^{(1)}_{\alpha,\beta}: \ y^2 = x^3 - 3\alpha x + (T + \frac{1}{T} - 2\beta) \tag{2}$$

for some constants α, β such that (α^3, β^2) is uniquely determined by X.

On the other hand, let $S = \mathrm{Km}(C_1 \times C_2)$ be the Kummer surface of the product of two elliptic curves C_1, C_2. Let j_1, j_2 denote their j-invariants. (The j-invariant is "classically" normalized so that $j = 1$ for the elliptic curve $y^2 = x^3 - x$ instead of $j = 1728$.) Let us write such an S as

$$S = S_{j_1, j_2}.$$

Proposition 3. *The Kummer surface $S = S_{j_1,j_2}$ admits an elliptic fibration $f : S \to \mathbf{P}^1_t$ which has two IV^*-fibres. Its defining equation is:*

$$F^{(2)}_{\alpha,\beta}: \ y^2 = x^3 - 3\alpha x + (t^2 + \frac{1}{t^2} - 2\beta) \tag{3}$$

where

$$\alpha = \sqrt[3]{j_1 j_2}, \ \beta = \sqrt{(1 - j_1)(1 - j_2)} \tag{4}$$

(the choice of the cube root or square root arbitrary).

Proposition 4. *The Kummer surface $S = S_{j_1,j_2}$ admits another elliptic fibration $f' : S \to \mathbf{P}^1_u$ with (at least) three singular fibres $II^*, I^*_{b_1}, I^*_{b_2}$, or II^*, I^*_0, IV^*. If we normalize the position of these fibres at the 3 points $u = \infty$, $u = \pm 2$ of the base curve $\mathbf{P}^1_u$, the defining equation is given by*

$$F^{(0)}_{\alpha,\beta}: \ y^2 = x^3 - 3\alpha(u^2 - 4)^2 x + (u - 2\beta)(u^2 - 4)^3 \tag{5}$$

where α, β are the same as before.

The map $\phi : S \to X$ in Th.1 corresponds to the map $F^{(2)} \to F^{(1)}$ defined by

$$(x, y, t) \mapsto (x, y, T), \quad T = t^2,$$

while the map $\psi : X \to S$ corresponds to the map $F^{(1)} \to F^{(0)}$ defined by

$$(x, y, T) \mapsto (X, Y, u), X = x(T - \frac{1}{T})^2, \ Y = y(T - \frac{1}{T})^3, u = T + \frac{1}{T}.$$

The automorphisms σ, τ are realized as follows:

$$\sigma : (x, y, t) \mapsto (x, y, -t), \quad \tau : (x, y, t) \mapsto (x, -y, 1/t).$$

We refer to [22] (cf.[8]) for the proof and the rest of arguments.

3 Mordell-Weil Lattices

To study the structure of the MWL (see [18]), we recall two general formulas from the theory of Mordell-Weil lattices [16]. Suppose E is an elliptic curve over $K = k(C)$ (C/k any curve) such that the associated elliptic surface $f : S \to C$ has at least one singular fibre. We identify the K-rational points $P \in E(K)$ with the sections $\sigma : C \to S$ and denote the image curve $Im(\sigma) \subset S$ by the symbol (P).

(1) Height formula: For any $P \in E(K)$, we have

$$\langle P, P \rangle = 2\chi(S) + 2(PO) - \sum_v contr_v(P), \tag{6}$$

where $\chi(S)$ is the arithmetic genus of S and (PO) denotes the intersection number of the section (P) and the zero-section (O) on the elliptic surface S ([16, Th.8.2]). The term $contr_v(P)$ is a local contribution at $v \in C$ where the fibre at v is reducible. Its value is determined by the type of reducible fibre at v and the component hit by the section (P), and it is given by ([16, (8.16)]).

(2) Determinant formula: Let M be the Mordell-Weil lattice $M = E(K)/E(K)_{tor}$. Then, by [16, Th.8.7], we have

$$\det M/|E(K)_{tor}|^2 = \pm \det NS(S)/\det V_S \tag{7}$$

where V_S denotes the trivial lattice of $f : S \to B$. For the convenience of the reader, we also write down the standard relation on the Picard number $\rho(S) = \text{rk } NS(S)$ and the MW-rank $r = \text{rk } M$ (see [16, (5.10)]):

$$r = \rho(S) - \text{rk } V_S. \tag{8}$$

4 Structure of MWL for some elliptic K3 with high rank

We consider the base change of $F^{(1)}$ via $T = t^n$:

$$F^{(n)}_{\alpha,\beta} : \; y^2 = x^3 - 3\alpha x + (t^n + \frac{1}{t^n} - 2\beta) \tag{9}$$

For every $0 < n \leq 6$, this equation defines an elliptic K3 surface. For simplicity, we assume that $k = \bar{\mathbb{Q}}$ and $j_1 \neq j_2$. Then, by [7] or [18], the Mordell-Weil rank is given by the formula:

$$r^{(n)}_{\alpha,\beta} = \text{rk}F^{(n)}_{\alpha,\beta}(k(t)) = \text{rkHom}(C_1, C_2) + \begin{cases} 4(n-1) & \text{if } n \leq 5, \\ 16 & \text{if } n = 6. \end{cases} \tag{10}$$

In the following, we look at the structure of the MWL and give explicit generators of $k(t)$-rational points in a special case: take $\alpha = 0, \beta = 0$ and let $n = 5$:

$$E = F^{(5)}_{0,0} : \ y^2 = x^3 + t^5 + \frac{1}{t^5}$$

This corresponds to the case: $j_1 = 0, j_2 = 1$, i.e.

$$C_1 : y^2 = x^3 - 1, \quad C_2 : y^2 = x^3 - x.$$

Since these elliptic curves are not isogenous in char 0, we have $\mathrm{Hom}(C_1, C_2) = 0$ and hence $E(k(t))$ has rank $r = r^{(5)}_{0,0} = 16$.

Set

$$s = t + \frac{1}{t}, \quad T = t^5, \quad w = T + \frac{1}{T}.$$

Then the elliptic curve E is actually defined over $k(w)$:

$$E : y^2 = x^3 + w,$$

and we want to study the MW-group $E(k(t))$. Observe that the field extension $k(t)/k(w)$ is a Galois extension with the Galois group $G = \langle \sigma, \tau \rangle$, where we denote now by

$$\sigma : t \to \zeta_5 \cdot t, \quad \tau : t \to \frac{1}{t}$$

(with ζ_5 a primitive 5-th root of unity). Thus G is the dihedral group of order 10, and the intermediate fields of $k(t)/k(w)$ correspond to the subgroups of G by Galois theory. In particular, $k(T)$ corresponds to $\langle \sigma \rangle$, while $k(s)$ to $\langle \tau \rangle$. We have

$$E(k(w)) = 0, \quad E(k(T)) = 0.$$

(The latter is actually a nontrivial consequence of $r^{(1)} = 0$ in our situation.) On the other hand, the MWL $E(k(s))$ is a lattice of rank 8, isomorphic to the root lattice E_8, because $E/k(s)$ defines a rational elliptic surface (RES) over the s-line, without reducible fibres:

$$E : y^2 = x^3 + s^5 - 5s^3 + 5s,$$

and the structure of such a MWL is wellknown to be E_8 (see [16]). Moreover this equation is a special case of the (E_8)-model treated in [17], where the method to determine the $k(s)$-rational points is given.

Now let $M = E(k(t))$ denote the MWL in question (note that $E(k(t))$ has no torsion). The sublattice $L = E(k(s))$ is isomorphic to $E_8[2]$, the scaling factor 2 being the degree of the extension $k(t)/k(s)$ (see [16, Prop.8.12]). Thus we have

$$M \supset L \cong E_8[2].$$

Now we consider the action of σ on M, and let $L' = L^\sigma$ denote the image of L in M. Then we claim that $L \cap L' = 0$ and $L + L'$ spans a sublattice of finite index in M. This is because we have $k(s) \cap k(s^\sigma) = k(w)$ by Galois theory and $E(k(w)) = 0$.

Theorem 5. *Let $M = E(k(t))$ be the MWL of the elliptic curve $E/k(t)$: $y^2 = x^3 + t^5 + 1/t^5$. Then M is an even integral lattice of rank 16 such that (i) $\det M = 5^4$, (ii)$M = L + L'$, (iii) Let $\{P_i(s)(1 \leq i \leq 8)\}$ be a basis of $L = E(k(s))$, then the 16 $k(t)$-rational points $\{P_i(t + 1/t), P_i(\zeta_5 t + 1/\zeta_5 t)\}$ form a set of generators of $E(k(t))$. (iv) The minimal norm (=minimal height) of M is 4, and the center density is equal to $1/5^2$. (v) The number of the minimal vectors (i.e. the kissing number of M) is at least 1200.*

Proof (Outline) Set $M' = L + L'$. This is the sublattice of M spanned by the 16 points above. The determinant $\det M'$ is equal to the height matrix $(\langle P_i, P_j \rangle)$ of these 16 points. Computing it using the height formula (6), we find that $\det M' = 5^4$. Letting ν be the index of M' in M, we have $\det M = 5^4/\nu^2$. The minimal norm of M is at least 4 by the height formula and actually it is equal to 4. Hence the center density of this 16-dimensional lattice M is:

$$\delta(M) = \nu/5^2.$$

As is remarked in [18], if $\nu > 1$, it would violate the sphere packing bound in dimension 16 ([2]), and hence we have $\nu = 1$. This implies $M = M'$, and all the assertions follow. *q.e.d.*

In carrying out the above outline, we have computed the explicit form of $P_i \in L = E(k(s))(i \leq 8)$ (cf. [17]). Each of these points is a "root" of the root lattice E_8 and takes the form:

$$P = (\frac{1}{u^2}(s^2 + as + b), \frac{1}{u^3}(s^3 + cs^2 + ds + e))$$

for suitable constants u, a, b, c, d, e, where c, d, e are determined by u, a, b. To give an idea, let us write down one point explicitly. Let

$$U = -1665 - 990\sqrt{3} - 784\sqrt{5} - 420\sqrt{15},$$

and let $u = u_0 = U^{1/12}$. We let $a = Au^6, b = BU$, where

$$A = 1460 + 843\sqrt{3} - 653\sqrt{5} - 377\sqrt{15},$$

and

$$B = 2041984 + 1178940\sqrt{3} - 913203\sqrt{5} - 527238\sqrt{15}.$$

This gives one of the points P_i, say P_1.

Corollary 6. *The splitting field of $E/\mathbf{Q}(s)$ (i.e. the smallest extension of* $\mathbf{Q}$ *over which a set of generators of $E(k(s))$ is defined) is given by*

$$\mathcal{K}_0 = \mathbf{Q}(\sqrt{3}, \sqrt{5}, \sqrt{-1})(u_0)$$

and the splitting field of $E/\mathbf{Q}(t)$ is equal to

$$\mathcal{K} = \mathcal{K}_0(\zeta_5) = \mathbf{Q}(\sqrt{3}, \sqrt{-1}, \zeta_5, u_0).$$

Remark Compare [25] for a different proof of Theorem 5(i). Note that the sum in (ii) $L + L'$ is not an orthogonal direct sum.

5 Elliptic modular surface of level 4 revisited

Let $f : S \to C$ denote the elliptic modular surface of level $N \geq 3$ ([13]). It is the universal family of elliptic curves with level N-structure, parametrized by the elliptic modular curve $C = C_N$ of level N. There are singular fibres of type I_N over the $t(N)$ cusps in C, where $t(N) = 1/2 \cdot N^2 \prod_{p|N}(1 - p^{-2})$.

Let E denote the generic fibre of f; it is an elliptic curve defined over $K = k(C)$, the field of elliptic modular functions of level N. Here the base field k is any field containing a primitive N-th root of unity. The MW-group $E(K)$ naturally contains the group of N-torsion points which is isomorphic to $(\mathbf{Z}/N\mathbf{Z})^{\oplus 2}$. Thus we have N^2 (disjoint) sections of $f : S \to C$.

Now, in char 0, we have proven ([13]) that these are the only sections. Geometrically this looks very natural. Indeed what else could appear as a section of such a universal family over the moduli space? So my initial guess was that the same phenomenon should hold true in any char $p > 0$ not dividing N.

For $N = 3$, this is true. It is known to Igusa [3] that S is given by the Hesse's normal form

$$E : X^3 + Y^3 + Z^3 - 3tXYZ = 0$$

and that $E(k(t))$ consists of the 9 base points of the linear pencil, the group of 3-torsion points. Another proof based on MWL is this: in this case, S is a rational elliptic surface with 4 singular fibres of type I_3. Thus the trivial lattice is $A_2^{\oplus 4} \subset E_8$, which implies that $E(k(t))$ is isomorphic to the quotient group $(\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$.

For $N = 4$, S is an elliptic K3 surface with 6 singular fibres of type I_4, which is defined by the equation:

$$E : y^2 = x(x - 4t^2)(x - (t^2 + 1)^2). \tag{11}$$

This is derived from the Legendre equation with $\lambda = (1/2(t+1/t))^2$ ([14]), by getting rid of the denominator in the coefficients.

In char 0, the Picard number $\rho = \text{rkNS}(S)$ is at most equal to the Hodge number $\rho \le h^{1,1}$ by Lefschetz-Hodge theory, and the rank of the trivial lattice (generated by the zero-section and the fibre components) is equal to the upper bound $h^{1,1} = 20$. This implies that $E(K)$ has rank 0 and $\cong (\mathbf{Z}/4\mathbf{Z})^{\oplus 2}$. (The same argument works for any level N.)

In char $p > 0$, however, we have only the weaker inequality $\rho \le b_2 = h^{1,1} + 2p_g$ where b_2 is the second Betti number and p_g is the geometic genus of the surface. For a K3 surface, we have $p_g = 1, b_2 = 22$. Nowadays everybody knows that there are K3 with $\rho = 22$, called *supersingular K3.* But at the time I worked on this problem (early 1970's), the situation was different. In connection with the arithmetic theory of surfaces, we have determined the zeta function of S ([13, Appendix]). Then the Tate conjecture (and the Birch-Swinnerton-Dyer conjecture) suggested that

$$\rho = \begin{cases} 20, & r = 0 \quad \text{if } p \equiv 1 \mod 4, \\ 22, & r = 2 \quad \text{if } p \equiv 3 \mod 4. \end{cases} \tag{12}$$

The former case is just as in char 0, which is easily settled. The latter case was against my initial guess above suggested by the geometric idea, and I tried for some time to disprove it (which would be a counterexample to the Tate conjecture), of course in vain.

About a year later, it occurred to me that, if it was impossible to disprove it, maybe I should try to find some new rational points. And I found, in the very special case $p = 3$, two independent points in $E(K)$ modulo torsion ([14, Prop.8]). Then I succeeded in fully proving (12), by showing that our surface S (the EMS of level 4) is a Kummer surface Km(A) for explicitly given abelian surface A (see [15]; this is my report at the "Conference Manifolds-Tokyo, 1973", where I learned from M.Artin that he and Swinnerton-Dyer proved the Tate conjecture for elliptic K3 surfaces.) Then came Artin's work [1] (1973), which introduced the notion of *Artin invariant* σ with the property that $\det NS(X) = p^{2\sigma}$ for X a supersingular K3 in char p; the elliptic modular surface S of level 4 played a nontrivial role there and it was shown to have $\sigma = 1$ if $p \equiv 3 \mod 4$. The most striking feature of Artin's theory is that supersingular K3's have 9-dimensional moduli (which sounded like a mathematical analogue of "superconductivity"!). The developments on the subject after it should be wellknown to most people and omitted (but to mention a few, Rudakov-Shafarevich, Ogus, etc).

Now let me present the update for EMS of level 4 in the supersingular case:

Theorem 7. *The Mordell-Weil lattice* $M = E(K)/E(K)_{tor}$ *of the EMS of level 4 in char* $p \equiv 3 \mod 4$ *is a rank 2 lattice with the height matrix* $\begin{pmatrix} p/4 & 0 \\ 0 & p/4 \end{pmatrix}$. *In other words, there exist two* $k(t)$*-rational points* $P_1, P_2 \in E(k(t))$ *with height* $p/4$ *such that* $\langle P_1, P_2 \rangle = 0$*, which generate* $E(k(t))$ *modulo torsion.*

Proof By the determinant formula (7), we have $\det M = (p/4)^2$, since we have $|E(K)_{tor}| = 4^2$, $\det NS(S) = p^2$ and $\det V_S = 4^6$ (6 singular fibres of type I_4). On the other hand, M is similar to the square lattice $\mathbf{Z}^{\oplus 2}$; this follows from the existence of order 4 automorphism of S acting faithfully on this lattice. Thus we prove Theorem 7 (cf. the recent paper [23] where a similar idea works).

Example (cf.[14]) Let $p = 3$ and let

$$P_1 = ((1-i)(t-i)t^2, (1+i)(t+1)(t-i)(t-1+i)t^2)$$

where $i = \sqrt{-1} \in \mathbf{F}_{p^2}$. Let P_2 be the conjugate of P_1 w.r.t. $i \to -i$. Then $\{P_1, P_2\}$ forms an orthogonal pair of rational points having height $p/4$.

By the height formula (6), one can check

$$\langle P_1, P_1 \rangle = 2 \cdot 2 + 2 \cdot 0 - \frac{3}{4} - \frac{3}{4} - \frac{3}{4} - 1 - 0 - 0 = \frac{3}{4}.$$

The local contribution $contr_v(P_1)$ is equal to $3/4$ for $v = \infty, 0, i$, to 1 for $v = -1$, and to 0 for $v = 1, -i$. Similarly for $\langle P_2, P_2 \rangle$ and $\langle P_1, P_2 \rangle$. The other point $P_3 = (t^4, -t^3(t^2-1))$ from [14] has height $p/2$ and fails to be a part of the orthogonal basis of M, but still $\{P_1, P_3\}$ generates M, as expected there.

It will be an amusing exercise in char p to play a similar game for other supersingular prime p, say $p = 7, 11, \ldots$. This was impossible at the time of [14], but now it should be possible. Theoretically the above Theorem 7 assures the existence of rational points with minimal height $p/4$, while the height formula will afford the technique for actual computation.

By the way, the height formula is useful even for the torsion points. In fact, the intersection diagram of all the N^2 torsion sections and all the irreducible components of singular fibres for the EMS of any level N can be described in terms of *a linear code* over the ring $\mathbf{Z}/N\mathbf{Z}$ (see [19]).

Finally let us call the attention of interested readers to the recent work by I.Shimada [12] and M.Schütt [11] on the transcendental lattices of singular K3 surfaces and the supersingular reduction lattices. Our results on the EMS of level 4 explained here and some other K3 in [20] give the concrete examples of their theory.

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