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1 Abstract

We denote by $\mathbf{G}$ the set of all lines in a complex projective space $\mathbb{P}^n$. For a hypersurface $X$ in $\mathbb{P}^n$, the set

$$Y_m = \{(p, L) \in \mathbb{P}^n \times \mathbf{G} \mid L \text{ and } X \text{ intersect at a point } p \text{ with the multiplicity } \geq m\}$$

form a projective variety, whose defining equations are given by using the higher derivative of the defining equation of $X$ (Proposition 2.1). The projective variety $Y_m$ is smooth for a general hypersurface $X$ (Theorem 2.2). A purpose of the research is to characterize some geometric properties of $X$ by using the Hodge structure of $Y_m$. In this report, we give a method to describe the Hodge cohomology of $Y_m$ by Jacobian rings, which is a new generalization of the theory of Jacobian ring for a hypersurface in $\mathbb{P}^n$ by Griffiths [2].

2 Varieties of intersection

We denote by $\mathbf{P} = \text{Grass}(V, n)$ the Grassmannian variety of all $n$-dimensional subspaces in $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and denote by $\mathcal{S}_\mathbf{P}$ (resp. $\mathcal{Q}_\mathbf{P}$) the universal sub (resp. quotient) bundle on $\mathbf{P}$. We have an exact sequence

$$0 \rightarrow \mathcal{S}_\mathbf{P} \rightarrow \mathcal{O}_\mathbf{P} \otimes V \rightarrow \mathcal{Q}_\mathbf{P} \rightarrow 0.$$

Then $\mathbf{P}$ is naturally identified with $\mathbb{P}^n$, and $\mathcal{Q}_\mathbf{P}$ is identified with $\mathcal{O}_{\mathbb{P}^n}(1)$. We denote by $\mathbf{G} = \text{Grass}(V, n - 1)$ the Grassmannian variety of all $(n - 1)$-dimensional subspaces in $V$, and denote by $\mathcal{S}_\mathbf{G}$ (resp. $\mathcal{Q}_\mathbf{G}$) the universal sub (resp. quotient) bundle on $\mathbf{G}$. We have an exact sequence

$$0 \rightarrow \mathcal{S}_\mathbf{G} \rightarrow \mathcal{O}_\mathbf{G} \otimes V \rightarrow \mathcal{Q}_\mathbf{G} \rightarrow 0.$$
We remark that a point in $G$ corresponds to a line in $P^n$. We denote by $\Gamma$ the subvariety of $P \times G$ defined as the zeros of the composition

$$p_2^* S_G \rightarrow O_{P \times G} \otimes V \rightarrow p_1^* Q_P,$$

where $p_1 : P \times G \rightarrow P$ (resp. $p_2 : P \times G \rightarrow G$) denotes the first (resp. second) projection. Then $\Gamma$ is the flag variety of all pairs of a line in $P^n$ and a point on the line. By the first projection $\phi = p_1|_\Gamma$, the subvariety $\Gamma$ is considered as the Grassmannian bundle ($P^{n-1}$-bundle)

$$\phi : \Gamma = \text{Grass}(S_P, n-1) \rightarrow P.$$

By the second projection $\pi = p_2|_\Gamma$, the subvariety $\Gamma$ is considered as the Grassmannian bundle ($P^1$-bundle)

$$\pi : \Gamma = \text{Grass}(Q_G, 1) \rightarrow G.$$

We denote by $Q_\phi$ the universal quotient bundle of the Grassmannian bundle $\phi$. We have exact sequences

$$0 \rightarrow \pi^* S_G \rightarrow \phi^* S_P \rightarrow Q_\phi \rightarrow 0$$

and

$$0 \rightarrow Q_\phi \rightarrow \pi^* Q_G \rightarrow \phi^* Q_P \rightarrow 0.$$

We introduce descending filtration

$$\text{Sym}^d \pi^* Q_G = \text{Fil}^0 \text{Sym}^d \pi^* Q_G \supset \cdots \supset \text{Fil}^{d+1} \text{Sym}^d \pi^* Q_G = 0$$

on the $d$-th symmetric product of $\pi^* Q_G$, where the subspace $\text{Fil}^m \text{Sym}^d \pi^* Q_G$ is defined as the image of the natural homomorphism

$$\text{Sym}^m Q_\phi \otimes \text{Sym}^{d-m} \pi^* Q_G \rightarrow \text{Sym}^d \pi^* Q_G.$$

For $F \in \text{Sym}^d V$, we denote by $X_F$ the hypersurface in $P$ defined as the zeros of the image of $F$ by the natural isomorphism

$$\text{Sym}^d V \simeq H^0(P, \text{Sym}^d Q_P),$$

we denote by $Y_{F, m}$ the subvariety in $\Gamma$ defined as the zeros of the image of $F$ by the natural homomorphism

$$\text{Sym}^d V \simeq H^0(\Gamma, \text{Sym}^d \pi^* Q_G) \rightarrow H^0(\Gamma, \text{Sym}^d \pi^* Q_G / \text{Fil}^m \text{Sym}^d \pi^* Q_G),$$

and we denote by $Z_F$ the subvariety in $G$ defined as the zeros of the image of $F$ by the natural isomorphism

$$\text{Sym}^d V \simeq H^0(G, \text{Sym}^d Q_G).$$
Then a point in $Z_F$ is corresponds to a line which is contained in $X_F$. Let $L$ be a line in $\mathbb{P}^n$, and let $p$ be a point on $L$. The fiber of the line bundle $\mathcal{Q}_\phi$ at the point on $\Gamma$ corresponding to the pair $(p, L)$ is naturally identified with the kernel of the restriction

$$H^0(L, \mathcal{O}_{\mathbb{P}^n}(1)|_L) \rightarrow H^0(p, \mathcal{O}_{\mathbb{P}^n}(1)|_p).$$

Hence, $L$ and $X_F$ intersect at $p$ with the multiplicity $\geq m$ if and only if the pair $(p, L)$ represent a point in $Y_{F,m}$. We have a diagram

\[
\begin{array}{cccc}
P & \phi & \Gamma & \pi \\
\uparrow & \uparrow & \uparrow & \uparrow \\
X_F & \phi(Y_{F,1}) & Y_{F,1} & \pi(Y_{F,1}) \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\vdots & \vdots & \vdots & \vdots \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\phi(Y_{F,d}) & Y_{F,d} & \pi(Y_{F,d}) \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\phi(Y_{F,d+1}) & Y_{F,d+1} & \pi(Y_{F,d+1}) = Z_F.
\end{array}
\]

The morphism $\pi|_{Y_{F,m}} : Y_{F,m} \rightarrow \pi(Y_{F,m})$ is finite for $1 \leq m \leq d$, and the morphism $\pi|_{Y_{F,d+1}} : Y_{F,d+1} \rightarrow Z_F$ is a $\mathbb{P}^1$-bundle.

We remark that the isomorphism

$$\text{Sym}^d \pi^* \mathcal{Q}_G / \text{Fil}^m \text{Sym}^d \pi^* \mathcal{Q}_G \simeq \text{Sym}^{d-m+1} \phi^* \mathcal{Q}_P \otimes \text{Sym}^m \pi^* \mathcal{Q}_G$$

is induced by the homomorphism

$$H^0(L, \mathcal{O}_{\mathbb{P}^n}(d)|_L) \rightarrow H^0(p, \mathcal{O}_{\mathbb{P}^n}(d - m + 1)|_p) \otimes H^0(L, \mathcal{O}_{\mathbb{P}^n}(m - 1)|_L);$$

$$A_1 \cdots A_d \rightarrow 1^{d!} \sum_{\sigma \in S_d} (A_{\sigma(1)} \cdots A_{\sigma(d-m+1)}|_p \otimes A_{\sigma(d-m+2)} \cdots A_{\sigma(d)}$$

where $A_i \in H^0(L, \mathcal{O}_{\mathbb{P}^n}(1)|_L)$, and $S_d$ denotes the permutation group of the index set $\{1, \ldots, d\}$.

Let $x_0, \ldots, x_n$ be a basis of $V$.

**Proposition 2.1.** The subvariety $Y_{F,m}$ in $\Gamma$ is defined as the zeros of the section

$$\sum_{0 \leq i_1, \ldots, i_{m-1} \leq n} \frac{\partial^{m-1} F}{\partial x_{i_1} \cdots \partial x_{i_{m-1}}} \otimes x_{i_1} \cdots x_{i_{m-1}}$$

$$\in \text{Sym}^{d-m+1} V \otimes \text{Sym}^{m-1} V \simeq H^0(\Gamma, \text{Sym}^{d-m+1} \phi^* \mathcal{Q}_P \otimes \text{Sym}^{m-1} \pi^* \mathcal{Q}_G).$$

The following theorem is proved by the similar way as [1], in which the corresponding results for the variety $Z_F$ is proved.
Theorem 2.2. Assume $1 \leq m \leq d + 1$.

1. If $m \leq n - 1$, then $Y_{F,m}$ is connected for any $F \in \text{Sym}^d V$.
2. If $m \leq 2n - 1$, then $Y_{F,m}$ is smooth of dimension $2n - m - 1$ for general $F \in \text{Sym}^d V$.

If $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then we can compute some topological invariants of $Y_{F,m}$. For example, if $m = d = 2n - 1$, then $\dim Y_{F,m} = 0$, and we can compute the number of the point of $Y_{F,m}$ by Schubert calculus;

$$
\begin{align*}
& m = d = 2n - 1 = 1 \implies \sharp Y_{F,m} = 1, \\
& m = d = 2n - 1 = 3 \implies \sharp Y_{F,m} = 9, \\
& m = d = 2n - 1 = 5 \implies \sharp Y_{F,m} = 575, \\
& m = d = 2n - 1 = 7 \implies \sharp Y_{F,m} = 99715,
\end{align*}
$$

for general $F$. It is similar to the case when $\dim Z_F = 0$;

$$
\begin{align*}
& d = 2n - 3 = 1 \implies \sharp Z_F = 1, \\
& d = 2n - 3 = 3 \implies \sharp Z_F = 9 \times 3 = 27, \\
& d = 2n - 3 = 5 \implies \sharp Z_F = 575 \times 5 = 2785, \\
& d = 2n - 3 = 7 \implies \sharp Z_F = 99715 \times 7 = 698005,
\end{align*}
$$

for general $F$. When $\dim Y_{F,m} = 1$ and $Y_{F,m}$ is connected, we can compute the genus of $Y_{F,m}$ for general $F$.

3 Jacobian rings

We denote by

$$
S = C[x_0, \ldots, x_n, z_0, \ldots, z_n] = \bigoplus_{p,q \in \mathbb{Z}} S^{p,q}
$$

the polynomial ring bi-graded by $\deg x_i = (1, 0)$ and $\deg z_j = (0, 1)$. We define homomorphisms $\delta$ and $\varepsilon$ by

$$
\delta : S^{p,q} \longrightarrow S^{p+1,q-1}; \ A \mapsto \sum_{i=0}^{n} \frac{\partial A}{\partial x_i} \cdot x_i
$$

and

$$
\varepsilon : S^{p,q} \longrightarrow S^{p-1,q+1}; \ A \mapsto \sum_{i=0}^{n} \frac{\partial A}{\partial x_i} \cdot z_i.
$$
For \( F \in \text{Sym}^d V \), we have a bi-homogeneous polynomial \( F_0 \in S^{d,0} \) by considering \( x_0, \ldots, x_n \) as a basis of \( V \). We set the bi-homogeneous polynomial \( F_k \) by

\[
F_k = \varepsilon^k(F_0) \in S^{d-k,k}
\]

for \( k \geq 1 \). We define the bi-graded ring \( S_{F,m} \) by

\[
S_{F,m} = S/(F_k; 0 \leq k \leq m-1),
\]

and we define the bi-graded ring \( R_{F,m} \) by

\[
R_{F,m} = S_{F,m-1}/\left( \frac{\partial F_{m-1}}{\partial x_i} \cdot x_j + (m-1) \frac{\partial F_{m-2}}{\partial x_i} \cdot z_j; 0 \leq i \leq n, 0 \leq j \leq n \right)
\]

for \( m \geq 1 \), where we set \( S_{F,0} = S \). Since

\[
\frac{1}{d} \sum_{i=0}^{n} \left( \frac{\partial F_{m-1}}{\partial x_i} \cdot x_i + (m-1) \frac{\partial F_{m-2}}{\partial x_i} \cdot z_i \right) = F_{m-1},
\]

the Jacobian ring \( R_{F,m} \) is a quotient ring of \( S_{F,m} \).

In the following, we describe the relation between the rings \( S_{F,m} \) and \( R_{F,m} \) and the variety \( Y_{F,m} \). Since the normal bundle \( N_{Y_{F,m}/\Gamma} \) of \( Y_{F,m} \) in \( \Gamma \) is isomorphic to \((\phi^\ast \text{Sym}^{d-m+1} Q_\mathbb{P} \otimes \pi^\ast \text{Sym}^{m-1} Q_\mathbb{G})|_{Y_{F,m}}\), using Lemma 4.5, we have the following proposition.

**Proposition 3.1.** If \( Y_{F,m} \) is smooth of dimension \( 2n - m - 1 \), then

\[
H^0(Y_{F,m}, N_{Y_{F,m}/\Gamma}) \simeq S_{F,m}^{d-m+1,m-1}
\]

for \( 1 \leq m \leq n - 1 \).

We denote by \( T_{\Gamma} \) (resp. \( T_{Y_{F,m}} \)) the tangent bundle of \( \Gamma \) (resp. \( Y_{F,m} \)). Then we have the exact sequences

\[
0 \longrightarrow O_{\Gamma} \longrightarrow \phi^\ast S_\mathbb{P}^\vee \otimes \pi^\ast Q_\mathbb{G} \longrightarrow T_{\Gamma} \longrightarrow 0
\]

and

\[
0 \longrightarrow T_{Y_{F,m}} \longrightarrow T_{\Gamma}|_{Y_{F,m}} \longrightarrow N_{Y_{F,m}/\Gamma} \longrightarrow 0,
\]

where \( S_\mathbb{P}^\vee \) denotes the dual of \( S_\mathbb{P} \). We remark that the composition

\[
V^\vee \otimes \pi^\ast Q_\mathbb{G} \longrightarrow \phi^\ast S_\mathbb{P}^\vee \otimes \pi^\ast Q_\mathbb{G} \longrightarrow T_{\Gamma} \longrightarrow N_{Y_{F,m}/\Gamma}
\]

induces the homomorphism

\[
V^\vee \otimes \nabla \simeq V^\vee \otimes H^0(\Gamma, \pi^\ast Q_\mathbb{G}) \longrightarrow H^0(Y_{F,m}, N_{Y_{F,m}/\Gamma}) \simeq S_{F,m}^{d-m+1,m-1};
\]

\[
x_i^\ast \otimes x_j \longmapsto \frac{(d-m+1)!}{d!} \left( \frac{\partial F_{m-1}}{\partial x_i} \cdot x_j + (m-1) \frac{\partial F_{m-2}}{\partial x_i} \cdot z_j \right),
\]

where \( x_0^\ast, \ldots, x_n^\ast \) denotes the dual basis of \( x_0, \ldots, x_n \). Using Lemma 4.6, we have the following theorem.
Theorem 3.2. If $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then there is a natural injective homomorphism

$$\rho : R^{d-m+1,m-1}_{F,m} \longrightarrow H^1(Y_{F,m}, T_{Y_{F,m}}),$$

and it is an isomorphism for $m \leq n - 3$.

We set the integers $\alpha(n,m,d,q)$ and $\beta(n,m,d,q)$ by

$$\begin{cases}
\alpha(n,m,d,q) = md - \frac{m(m-1)}{2} - n - 2 + q(d - m + 1), \\
\beta(n,m,d,q) = \frac{m(m-1)}{2} - n + q(m - 1).
\end{cases}$$

Since the canonical bundle $\Omega_{Y_{F,m}}^{2n-m-1}$ of $Y_{F,m}$ is isomorphic to $(\phi^* \text{Sym}^{\alpha(n,m,d,0)} Q_\Phi \otimes \text{Sym}^{\beta(n,m,d,0)} Q_\phi)|_{Y_{F,m}}$, using Lemma 4.5, we have the following theorem.

Theorem 3.3. If $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then there is a natural injective homomorphism

$$\gamma_0 : \text{Ker} \left( S^\alpha_{F,m}(n,m,d,0), \beta(n,m,d,0) - \delta \left( S^\alpha_{F,m}(n,m,d,0)+1, \beta(n,m,d,0)-1 \right) \right) \longrightarrow H^0(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-1}),$$

and it is an isomorphism for $m \leq n - 2$ or $m = n - 1 \leq 5$.

Here we remark that

$$S^{\alpha(n,m,d,0), \beta(n,m,d,0)}_{F,m} = \text{Ker} \left( S^\alpha_{F,m}(n,m,d,0), \beta(n,m,d,0) - \delta \left( S^\alpha_{F,m}(n,m,d,0)+1, \beta(n,m,d,0)-1 \right) \right)$$

for $\frac{m(m-1)}{2} \leq n$.

The following theorem is proved by the similar way as Theorem 3.2, by using the exact sequence

$$0 \longrightarrow \Omega_{Y_{F,m}}^{2n-m-2} \longrightarrow T_{\Gamma}|_{Y_{F,m}} \otimes \Omega_{Y_{F,m}}^{2n-m-1} \longrightarrow N_{Y_{F,m}/\Gamma} \otimes \Omega_{Y_{F,m}}^{2n-m-1} \longrightarrow 0.$$

Theorem 3.4. If $\frac{m(m-1)}{2} \leq n$, and $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then there is a natural injective homomorphism

$$\gamma_1 : R^\alpha_{F,m}(n,m,d,1), \beta(n,m,d,1) \longrightarrow H^1(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-2}),$$

and it is an isomorphism for $m \leq n - 3$.

Theorem 3.5. If $\frac{m(m-1)}{2} \leq n$, and $Y_{F,m}$ is smooth of dimension $2n - m - 1$, then the diagram

$$\begin{array}{ccc}
R^{d-m+1,m-1}_{F,m} \otimes S^{\alpha(n,m,d,0), \beta(n,m,d,0)}_{F,m} & \longrightarrow & R^\alpha_{F,m}(n,m,d,1), \beta(n,m,d,1) \\
\rho \otimes \gamma_0 & \downarrow & \gamma_1 \\
H^1(Y_{F,m}, T_{Y_{F,m}}) \otimes H^0(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-1}) & \longrightarrow & H^1(Y_{F,m}, \Omega_{Y_{F,m}}^{2n-m-2})
\end{array}$$
commutes, where the homomorphism $\mu$ is defined by the multiplication of the ring $R_{F,m}$, and the homomorphism $\nu$ is defined by the composition of the cup product and the contraction.

4 Calculation of cohomology

In this section, we enumerate several lemmas, which is used in the proof of theorems in Section 3. For simplicity of notations, we set the invertible sheaf $\mathcal{O}_\Gamma(p,q)$ on $\Gamma$ by

$$
\mathcal{O}_\Gamma(p,q) = \begin{cases} 
\text{Sym}^p \phi^* \mathcal{Q}_p \otimes \text{Sym}^q \mathcal{Q}_q \quad (p \geq 0, \ q \geq 0), \\
\text{Sym}^p \phi^* \mathcal{Q}_p \otimes \text{Sym}^{-q} \mathcal{Q}_q' \quad (p \geq 0, \ q < 0), \\
\text{Sym}^{-p} \phi^* \mathcal{Q}_p' \otimes \text{Sym}^q \mathcal{Q}_q \quad (p < 0, \ q \geq 0), \\
\text{Sym}^{-p} \phi^* \mathcal{Q}_p' \otimes \text{Sym}^{-q} \mathcal{Q}_q' \quad (p < 0, \ q < 0), 
\end{cases}
$$

and we set $\mathcal{Q}_G^r = \pi^* \text{Sym}^r \mathcal{Q}_G$ for $r \geq 0$. For a sheaf $\mathcal{E}$ of $\mathcal{O}_\Gamma$-modules, we set $\mathcal{E}(p,q) = \mathcal{E} \otimes \mathcal{O}_\Gamma(p,q)$.

Lemma 4.1. Assume $r \geq 0$. $H^0(\Gamma, \mathcal{Q}_G^r(p,q)) = \text{Ker} \ (\delta^{r+1} : S^{p,q+r} \rightarrow S^{p+r+1,q-1})$.

Lemma 4.2. Assume $q \leq 0$ and $r \geq 0$.

1. $H^j(\Gamma, \mathcal{Q}_G^r(p,q)) = 0$ for $1 \leq j \leq n-2$.
2. When $n \geq 2$, if $q \geq -n+1$ or $p+r \leq -2$, then $H^{n-1}(\Gamma, \mathcal{Q}_G^r(p,q)) = 0$.

Lemma 4.3. Assume $q \leq 0$.

1. $H^j(\Gamma, T_\Gamma(p,q)) = 0$ for $1 \leq j \leq n-3$.
2. When $n \geq 3$, if $q \geq -n+1$ or $p \leq -2$, then $H^{n-2}(\Gamma, T_\Gamma(p,q)) = 0$.

Lemma 4.4. Assume $q \leq 0$ and $r \geq 0$.

1. $H^1(Y_{F,m}, \mathcal{Q}_G^r(p,q)|_{Y_{F,m}}) = 0$ for $1 \leq m \leq n-3$.
2. If $q \geq \frac{n(n-7)}{2} + 4$ or $p+r \leq (n-2)d - \frac{n(n-5)}{2} - 5$, then

$$
H^1(Y_{F,n-2}, \mathcal{Q}_G^r(p,q)|_{Y_{F,n-2}}) = 0.
$$

Lemma 4.5. Assume $r \geq 0$.

1. $H^0(Y_{F,m}, \mathcal{Q}_G^r(p,q)|_{Y_{F,m}}) \simeq \text{Ker} \ (\delta^{r+1} : S^{p,q+r} \rightarrow S^{p+r+1,q-1})$

for $1 \leq m \leq n-2$.
2. If $\min \{q,0\} \geq \frac{n(n-5)}{2} + 2$ or $p+r + \max \{q,0\} \leq (n-1)d - \frac{n(n-3)}{2} - 3$, then

$$
H^0(Y_{F,n-1}, \mathcal{Q}_G^r(p,q)|_{Y_{F,n-1}}) \simeq \text{Ker} \ (\delta^{r+1} : S^{p,q+r} \rightarrow S^{p+r+1,q-1}).
$$
Lemma 4.6. Assume $q \leq 0$.

1. $H^1(Y_{F,m}, T_Y(p, q)|_{Y_{F,m}}) = 0$ for $1 \leq m \leq n - 4$.
2. If $q \geq \frac{n(n-9)}{2} + 7$ or $p \leq (n-3)d - \frac{n(n-7)}{2} - 8$, then
   \[ H^1(Y_{F,n-3}, T_Y(p, q)|_{Y_{F,n-3}}) = 0. \]

5 The case $n = 3$ and $m = 3$

In this section, we consider a hypersurface $X_F$ in $\mathbb{P}^3$.

Proposition 5.1. If the variety $Y_{F,3}$ is smooth of dimension 2, then the morphism $\phi|_{Y_{F,3}} : Y_{F,3} \to X_F$ is the double covering branched along $B_F$, where $B_F$ is the divisor on $X_F$ defined by the equation

\[ \det \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{0 \leq i, j \leq 3} = 0. \]

By the results in Section 3, we have natural injective homomorphisms

\[ \rho : R^{d-2,2}_{F,3} \to H^1(Y_{F,3}, T_{Y_{F,3}}), \]

\[ \gamma_0 : S^{3d-8,0}_{F,3} \to H^0(Y_{F,3}, \Omega^2_{Y_{F,3}}) \]

and

\[ \gamma_1 : R^{4d-10,2}_{F,3} \to H^1(Y_{F,3}, \Omega^1_{Y_{F,3}}). \]

Proposition 5.2. If $d \geq 3$, then the homomorphism

\[ R^{d,0}_{F,1} \xrightarrow{\epsilon^2} R^{d-2,2}_{F,3} \to \text{Hom}_\mathbb{C}(S^{3d-8,0}_{F,3}, R^{4d-10,2}_{F,3}) \]

is injective for general $F \in S^{d,0}$.

We consider the period map

\[ \psi : M \to W; \quad [X_F] \mapsto [H^2(Y_{F,3})], \]

where $M$ denotes the set of isomorphism classes of hypersurfaces $X_F$ in $\mathbb{P}^3$ such that $Y_{F,3}$ is smooth, and $W$ denotes the set of isomorphism classes of Hodge structures of weight 2. By Proposition 5.2 and Theorem 3.4, the differential $d\psi$ of the period map $\psi$ at a general point in $M$ is injective, where we remark that the sets $M$ and $W$ have geometric structure. Now we have a natural question of Torelli type.

Question 5.3. For smooth surfaces $X_{F_1}$ and $X_{F_2}$ in $\mathbb{P}^3$, if there is an isomorphism $H^2(Y_{F_1,3}) \simeq H^2(Y_{F_2,3})$ as Hodge structures, then is there an isomorphism $X_{F_1} \simeq X_{F_2}$ as algebraic varieties?
5.1 The case \( d = 3 \)

In the following, we assume that \( d = 3 \). If \( Y_{F,3} \) is smooth, then \( Y_{F,3} \) is a minimal algebraic surface with the geometric genus \( p_g = 4 \), the irregularity \( q = 0 \) and the square of the first chern class \( c_1^2 = 6 \). Such algebraic surfaces are classified by Horikawa, and \( Y_{F,3} \) is called of type Ib in [3].

**Proposition 5.4.** For \( F \in S^{3,0} \), the cubic surface \( X_F \) is smooth if and only if \( Y_{F,3} \) is a smooth surface.

If \( X_F \) is a smooth cubic surface, then \( X_F \) contains 27 lines, which means that \( \#Z_F = 27 \). Hence \( Y_{F,4} \) is a disjoint union of 27 rational curves, which are \((−3)\)-curves in \( Y_{F,3} \).

**Proposition 5.5.** If \( X_F \) is a smooth cubic surface, then \( B_F \) has at most nodes as its singularities. A point \( p \in X_F \) is a node of \( B_F \) if and only if there are three lines in \( X_F \) which contains the point \( p \).

Since the morphism \( \phi|_{Y_{F,3}} : Y_{F,3} \to \mathbb{P}^3 \) is the canonical map for \( d = 3 \), we have the following proposition.

**Proposition 5.6.** For smooth cubic surfaces \( X_{F_1} \) and \( X_{F_2} \), there is an isomorphism \( X_{F_1} \cong X_{F_2} \) if and only if there is an isomorphism \( Y_{F_1,3} \cong Y_{F_2,3} \).

In the case when \( d = 3 \), the Hodge structure \( H^2(X_F) \) is trivial, but the Hodge structure \( H^2(Y_{F,3}) \) is not trivial. Hence the Question 5.3 is particularly interesting in this case.

**References**


**Graduate School of Science, Osaka University,**
**Toyonaka, Osaka, 560-0043, Japan**
**E-mail address: atsushi@math.sci.osaka-u.ac.jp**