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BIVARIANT THEORIES AND ALGEBRAIC COBORDISM

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1. Introduction

W. Fulton and R. MacPherson introduced the notion of bivariant theory as a *categorical framework for the study of singular spaces*, which is the title of their AMS Memoir book [FM]. It is a sophisticated unification of covariant functors and contravariant functors. The main objective of [FM] (also see [Fu]) is bivariant-theoretic Riemann–Roch's or bivariant analogues of various theorems of Grothendieck–Riemann–Roch type.

V. Voevodsky introduced algebraic cobordism (now called *higher algebraic cobordism*), which was used in his proof of Milnor's conjecture [Voe]. In an attempt to understand Voevodsky's algebraic cobordism, M. Levine and F. Morel constructed *a universal oriented cohomology theory*, which they also call *algebraic cobordism*, and have investigated furthermore (see [L1, L2, LM1, LM2, LM3] and also see [Mer] for a condensed review). It was also their attempt to understand an algebro-geometric analogue of *(complex) oriented cohomology theory* of D. Quillen on the category of differential manifolds [Qui]. Recently M. Levine and R. Pandharipande [LP] gave another equivalent construction of the algebraic cobordism via what they call "double point degeneration" and they found a nice application of algebraic cobordism in the Donaldson–Thomas theory of 3-folds.

We have been trying to unify Fulton–MacPherson's bivariant theory and Levine–Morel's algebraic cobordism, or more precisely, to capture algebraic cobordism as a special case of *a bivariant algebraic cobordism.* One motivation for this work is that the author has been working on characteristic classes of singular spaces (e.g., see [BSY], [SY] and [Y1] and references therein) and that bivariant theory and the abelian group generated by isomorphism classes of morphisms, a basic ingredient of Levine–Morel's algebraic cobordism, play important roles in the above works.

In this note we make a quick review of Fulton–MacPherson's bivariant theory and we give two theorems of [Y2], which is a basis of our further works [Y3, Y4].

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2. Fulton–MacPherson's bivariant theory

Let $\mathcal{V}$ be a category which has a final object pt and on which the fiber product or fiber square is well-defined. Also we consider a class of maps, called "confined maps" (e.g., proper maps, projective maps, in algebraic geometry), which are closed under composition and base change and contain all the identity maps, and a calss of fiber squares, called "independent squares" (or "confined squares", e.g., "Tor-independent" in algebraic geometry, a fiber square with some extra conditions required on morphisms of the square), which satisfy the following:

(i) if the two inside squares in

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$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

or

$$\begin{tikzcd} X' \arrow[r, "h''"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h'"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "h"'] & Z \end{tikzcd}$$

are independent, then the outside square is also independent,
(ii) any square of the following forms are independent:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_X"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $f : X \to Y$ is any morphism.

A bivariant theory $\mathbb{B}$ on a category $\mathcal{V}$ with values in the category of (graded) abelian groups is an assignment to each morphism

$$X \xrightarrow{f} Y$$

in the category $\mathcal{V}$ a (graded) abelian group (in the rest of the paper we ignore the grading)

$$\mathbb{B}(X \xrightarrow{f} Y)$$

which is equipped with the following three basic operations:

Product operations: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.

Pushforward operations: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f confined, the pushforward operation

$$f_* : \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.

Pullback operations: For an independent square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

the pullback operation

$$g^* : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined.

And these three operations are required to satisfy the seven compatibility axioms (see [FM, Part I, §2.2] for details):
(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,
(B-6) pushforward and pullback commute, and
(B-7) projection formula.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathcal{V}$. Then a *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism $X \to Y$ in the category $\mathcal{V}$, which preserves the above three basic operations:

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_*\alpha) = f_*\gamma(\alpha)$, and
(iii) $\gamma(g^*\alpha) = g^*\gamma(\alpha)$.

For more details of interesting geometric and/or topological examples of bivariant theories (e.g., bivariant theory of constructible functions, bivariant homology theory, bivariant K-theory, etc.,) and Grothendieck transformations among bivariant theories, see [FM].

A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense:
$\mathbb{B}_*(X) := \mathbb{B}(X \to pt)$ become a covariant functor for *confined* morphisms and
$\mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{id} X)$ become a contravariant functor for *any* morphisms.
And a Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ induces natural transformations $\gamma_* : \mathbb{B}_* \to \mathbb{B}'_*$ and $\gamma^* : \mathbb{B}^* \to \mathbb{B}'^*$.

As to the grading, $\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \xrightarrow{id} X)$ and $\mathbb{B}^j(X) := \mathbb{B}^j(X \xrightarrow{id} X)$.

In the rest of the paper we assume that our bivariant theories are commutative (see [FM, §2.2]), i.e., if whenever both

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

are independent squares, then for $\alpha \in \mathbb{B}(X \xrightarrow{f} Z)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$

(Note: if $g^*(\alpha) \bullet \beta = (-1)^{\deg(\alpha)\deg(\beta)} f^*(\beta) \bullet \alpha$, then it is called skew-commutative.)

Another assumption in the rest of the paper, which is not in Fulton–MacPherson's bivariant theory, is the following additivity:

$$\mathbb{B}(X \coprod Y \xrightarrow{f} Z) = \mathbb{B}(X \xrightarrow{f|_X} Z) \oplus \mathbb{B}(Y \xrightarrow{f|_Y} Z).$$

When we want to emphasize this additivity, we call such a bivariant theory an *additive bivariant theory*.

Definition 2.1. ([FM, 2.6.2 Definition, Part I]) Let $\mathcal{S}$ be a class of maps in $\mathcal{V}$, which is closed under compositions and containing all identity maps. Suppose that to each $f : X \to Y$ in $\mathcal{S}$ there is assigned an element $\theta(f) \in \mathbb{B}(X \xrightarrow{f} Y)$ satisfying that

(i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$ for all $f : X \to Y, g : Y \to Z \in \mathcal{S}$ and

(ii) $\theta(\mathrm{id}_X) = 1_X$ for all X with $1_X \in \mathbb{B}^*(X) := B(X \xrightarrow{\mathrm{id}_X} X)$ the unit element.

Then $\theta(f)$ is called a *canonical orientation* of f.

Note that such a canonical orientation makes the covariant functor $\mathbb{B}_*(X)$ a contravariant functor for morphisms in $\mathcal{S}$, and also makes the contravariant functor $\mathbb{B}^*$ a covariant functor for morphisms in $\mathcal{C} \cap \mathcal{S}$: Indeed,

(*) for a morphism $f : X \to Y \in \mathcal{S}$ and the canonical orientation θ on $\mathcal{S}$ the following *Gysin homomorphism*

$$f^! : \mathbb{B}_*(Y) \to \mathbb{B}_*(X) \quad \text{defined by} \quad f^!(\alpha) := \theta(f) \bullet \alpha$$

is *contravariantly functorial*. And

(**) for a fiber square (which is an independent square by hypothesis)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

where $f \in \mathcal{C} \cap \mathcal{S}$, the following *Gysin homomorphism*

$$f_! : \mathbb{B}^*(X) \to \mathbb{B}^*(Y) \quad \text{defined by} \quad f_!(\alpha) := f_*(\alpha \bullet \theta(f))$$

is *coavariantly functorial*. The notation should carry the information of $\mathcal{S}$ and the canonical orientation θ, but it will be usually omitted if it is not necessary to be mentioned. Note that the above conditions (i) and (ii) of Definition (2.1) are certainly necessary for the above Gysin homomorphisms to be functorial.

Definition 2.2. (i) Let $\mathcal{S}$ be another class of maps in $\mathcal{V}$, called "specialized maps" (e.g., smooth maps in algebraic geometry), which is closed under composition and under base change and containing all identity maps. Let $\mathbb{B}$ be a bivariant theory. If $\mathcal{S}$ has canonical orientations in $\mathbb{B}$, then we say that $\mathcal{S}$ is canonical $\mathbb{B}$-orientable and an element of $\mathcal{S}$ is called a canonical $\mathbb{B}$-orientable morphism. (Of course $\mathcal{S}$ is also a class of confined maps, but since we consider the above extra condition of $\mathbb{B}$-orientability on $\mathcal{S}$, we give a different name to $\mathcal{S}$.)

(ii) Let $\mathcal{S}$ be as in (i). Let $\mathbb{B}$ be a bivariant theory and $\mathcal{S}$ be canonical $\mathbb{B}$-orientable. Furthermore, if the orientation θ on $\mathcal{S}$ satisfies that for a fiber square with $f \in \mathcal{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

the following condition holds

$$\theta(f') = g^*\theta(f),$$

(which means that the orientation θ preserves the pullback operation), then we call θ a *nice canonical orientation* and say that $\mathcal{S}$ is *nice canonical* $\mathbb{B}$*-orientable* and an element of $\mathcal{S}$ is called *a nice canonical* $\mathbb{B}$*-orientable morphism* .

Proposition 2.3. *Let* $\mathbb{B}$ *be a bivariant theory and let* $\mathcal{S}$ *be as above.*

(1) Define the natural exterior product

$$\times : \mathbb{B}(X \to pt) \times \mathbb{B}(Y \xrightarrow{\pi_Y} pt) \to \mathbb{B}(X \times Y \to pt)$$

by

$$\alpha \times \beta := \pi_Y^* \alpha \bullet \beta.$$

Then the covariant functor $\mathbb{B}_*$ for confined morphisms and the contravariant functor $\mathbb{B}_*$ for morphisms in $\mathcal{S}$ are both compatible with the exterior product, i.e., for confined morphisms $f : X \to X'$, $g : Y \to Y'$,

$$(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$$

and for morphisms $f : X \to X'$, $g : Y \to Y'$ in $\mathcal{S}$,

$$(f \times g)^!(\alpha' \times \beta') = f^!\alpha' \times g^!\beta'.$$

(2) Similarly, define the natural exterior product

$$\times : \mathbb{B}(X \xrightarrow{\mathrm{id}_X} X) \times \mathbb{B}(Y \xrightarrow{\mathrm{id}_Y} Y) \to \mathbb{B}(X \times Y \xrightarrow{\mathrm{id}_{X \times X}} X \times X)$$

by

$$\alpha \times \beta := p_1{}^*\alpha \bullet p_2{}^*\beta$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections.

Then the contravariant functor $\mathbb{B}^*$ for any morphisms and the covariant functor $\mathbb{B}^*$ for morphisms in $C \cap \mathcal{S}$ are both compatible with the exterior product, i.e., for any morphisms $f : X \to X'$, $g : Y \to Y'$,

$$(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta$$

and for morphisms $f : X \to X'$, $g : Y \to Y'$ in $C \cap \mathcal{S}$,

$$(f \times g)_!(\alpha' \times \beta') = f_!\alpha' \times g_!\beta'.$$

Proposition 2.4. *Let $\mathbb{B}$ be a bivaraint theory on a category $\mathcal{V}$ with a class C of confined morphisms. Let $\mathcal{S}$ be a class of nice $\mathbb{B}$-orientable morphisms. Then for any independent square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

with $f \in \mathcal{S}$ and $g \in C$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}_*(Y') & \xrightarrow{f'^*} & \mathbb{B}_*(X') \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g'_*} \\ \mathbb{B}_*(Y) & \xrightarrow[f^*]{} & \mathbb{B}_*(X). \end{array}$$

3. A "UNIVERSAL" BIVARIANT THEORY

The following theorem is motivated by the construction of Levine–Morel's algebraic cobordism.

Theorem 3.1. *Let $\mathcal{V}$ be a cateogry with a class C of confined morphisms, a class of independent squares and a class $\mathcal{S}$ of specialized maps. We define*

$$\mathbb{M}^{C}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

to be the free abelian group generated by the set of isomorphism classes of confined morphisms $h : W \to X$ such that the composite of h and f is a specialized map:

$$h \in C \quad and \quad f \circ h : W \to Y \in \mathcal{S}.$$

(1) The association $\mathbb{M}^{C}_{\mathcal{S}}$ is a bivariant theory if the three operations are defined as follows:

<u>Product operations:</u> *For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation*

$$\bullet : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \otimes \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{gf} Z)$$

is defined by

$$\left(\sum_V m_V[V \xrightarrow{h_V} X]\right) \bullet \left(\sum_W n_W[W \xrightarrow{k_W} Y]\right) := \sum_{V,W} m_V n_W [V' \xrightarrow{h_V \circ k_W''} X],$$

where we consider the following fiber squares

$$\begin{tikzcd}
V' \arrow[r, "h'_V"] \arrow[d, "k_W''"'] & X' \arrow[r, "f'"] \arrow[d, "k_W'"'] & W \arrow[d, "k_W"'] & \\
V \arrow[r, "h_V"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z.
\end{tikzcd}$$

<u>Pushforward operations:</u> *For morphisms $f : X \to Y$ and $g : Y \to Z$ with f <u>confined</u>, the pushforward operation*

$$f_* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{gf} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z)$$

is defined by

$$f_*\left(\sum_V n_V[V \xrightarrow{h_V} X]\right) := \sum_V n_V[V \xrightarrow{f \circ h_V} Y].$$

<u>Pullback operations:</u> *For an independent square*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y,
\end{tikzcd}$$

the pullback operation

$$g^* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X' \xrightarrow{f'} Y')$$

is defined by

$$g^*\left(\sum_V n_V[V \xrightarrow{h_V} X]\right) := \sum_V n_V[V' \xrightarrow{h_V'} X'],$$

where we consider the following fiber squares:

$$\begin{tikzcd}
V' \arrow[r, "g''"] \arrow[d, "h_V'"'] & V \arrow[d, "h_V"] \\
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y.
\end{tikzcd}$$

(2) Let $\mathcal{BT}$ be a class of bivariant theories $\mathbb{B}$ on the same category $\mathcal{V}$ with a class $\mathcal{C}$ of confined morphisms, a class of independent squares and a class $\mathcal{S}$ of specialized maps. Let $\mathcal{S}$ be <u>nice</u> canonical $\mathbb{B}$-orientable for any bivariant theory $\mathbb{B} \in \mathcal{BT}$. Then, for each bivariant theory $\mathbb{B} \in \mathcal{BT}$ there exists a unique Grothendieck transformation

$$\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{B}$$

such that for a specialized morphism $f : X \to Y \in \mathcal{S}$ *the homomorphism* $\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$ *satisfies the normalization condition that*

$$\gamma_{\mathbb{B}}([X \xrightarrow{\mathrm{id}_X} X]) = \theta_{\mathbb{B}}(f).$$

4. ORIENTED BIVARIANT THEORIES

Definition 4.1. Let $\mathbb{B}$ be a bivariant theory on a category. Let $\mathcal{L}$ be a class of morphisms in $\mathcal{V}$, called "line bundles" (e.g., line bundles in geometry), which are closed under base change. As in the theory of bundles, for a line bundle $L \to X$, we simply denote it by the source object L, unless some confusion is possible. For a line bundle $L \in \mathcal{L}$, the *"first Chern class operator" on* $\mathbb{B}$ *associated to the "line bundle"* L, denoted by $\tilde{c}_1(L)$, is an endomorphism

$$\tilde{c}_1(L) : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$$

which satisfies the following properties:

(O-1) identity: If two line bundles $L \to X$ and $M \to X$ are isomorphic, i.e., there exists an isomorphism $L \cong M$ over X, then we have

$$\tilde{c}_1(L) = \tilde{c}_1(M) : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y).$$

(O-2) commutativity: Let $L \to X$ and $L' \to X$ be two line bundles over X, then we have

$$\tilde{c}_1(L) \circ \tilde{c}_1(L') = \tilde{c}_1(L') \circ \tilde{c}_1(L) : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y).$$

(O-3) compatibility with product: For morphisms $f : X \to Y$ and $g : Y \to Z$, $\alpha \in \mathbb{B}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$, a line bundle $L \to X$ and a line bundle $M \to Y$

$$\tilde{c}_1(L)(\alpha \bullet \beta) = \tilde{c}_1(L)(\alpha) \bullet \beta, \quad \tilde{c}_1(f^*M)(\alpha \bullet \beta) = \alpha \bullet \tilde{c}_1(M)(\beta)$$

(O-4) compatibility with pushforward: With f being confined

$$f_*(\tilde{c}_1(f^*M)(\alpha)) = \tilde{c}_1(M)(f_*\alpha).$$

(O-5) compatibility with pullback: For an independent square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

$$g^*(\tilde{c}_1(L)(\alpha)) = \tilde{c}_1(g'^*L)(g^*\alpha).$$

The above first Chern class operator is called an *"orientation"* and a bivariant theory equipped with such a first Chern class operator is called an *oriented bivariant theory*, denoted by $\mathbb{OB}$. An *oriented Gorthendieck transformation* between two oriented bivariant theories is a Grothendieck transformation which preserves or is compatible with the first Chern class operator.

Let us consider a morphism $h_V : V \to X$ equipped with finitely many line bundles over the source variety V of the morphism h_V:

$$(V \xrightarrow{h_V} X; L_1, L_2, \cdots, L_r)$$

with L_i being a line bundle over V. This family is called a *cobordism cycle over* X [LM3, Mer]. Then $(V \xrightarrow{h_V} X; L_1, L_2, \cdots, L_{r'})$ is isomorphic to $(W \xrightarrow{h_W} X; M_1, M_2, \cdots, M_r)$ if and only if h_V and h_W are isomorphic, i.e., there is an isomorphism $g : V \cong W$ over X, there is a bijection $\sigma : \{1, 2, \cdots, r\} \cong \{1, 2, \cdots, r'\}$ (so that $r = r'$) and there are isomorphisms $L_i \cong g^* M_{\sigma(i)}$ for every i.

Theorem 4.2. *(A universal oriented bivariant theory) Let $\mathcal{V}$ be a cateogry with a class $\mathcal{C}$ of confined morphisms, a class of independent squares, a class $\mathcal{S}$ of specialized morphisms and a class $\mathcal{L}$ of line bundles. We define*

$$\mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

to be the free abelian group generated by the set of isomorphism classes of cobordism cycles over X

$$[V \xrightarrow{h} X; L_1, L_2, \cdots, L_r]$$

such that the composite of h and f

$$f \circ h : W \to Y \in \mathcal{S}.$$

(1) The association $\mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}$ becomes an oriented bivariant theory if the four operations are defined as follows:

<u>Orientation</u>: For a morphism $f : X \to Y$ and a line bundle $L \to X$, the first Chern class operator

$$\tilde{c}_1(L) : \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

is defined by

$$\tilde{c}_1(L)([V \xrightarrow{h_V} X; L_1, L_2, \cdots, L_r]) := [V \xrightarrow{h_V} X; L_1, L_2, \cdots, L_r, (h_V)^*L].$$

<u>Product operations</u>: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \otimes \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z) \to \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{gf} Z)$$

is defined as follows: The product on generators is defined by

$$[V \xrightarrow{h_V} X; L_1, \cdots, L_r] \bullet [W \xrightarrow{k_W} Y; M_1, \cdots, M_s]$$
$$:= [V' \xrightarrow{h_V \circ k_W''} X; {k_W''}^* L_1, \cdots, {k_W''}^* L_r, (f' \circ h_V')^* M_1, \cdots, (f' \circ h_V')^* M_s],$$

and it extends bilinearly. Here we consider the following fiber squares

$$\begin{tikzcd}
V' \arrow[r, "h'_V"] \arrow[d, "k_W''"'] & X' \arrow[r, "f'"] \arrow[d, "k_W'"'] & W \arrow[d, "k_W"'] & \\
V \arrow[r, "h_V"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z.
\end{tikzcd}$$

<u>Pushforward operations</u>: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f <u>confined</u>, the pushforward operation

$$f_* : \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{gf} Z) \to \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z)$$

is defined by

$$f_*\left(\sum_V n_V [V \xrightarrow{h_V} X; L_1, \cdots, L_r]\right) := \sum_V n_V [V \xrightarrow{f \circ h_V} Y; L_1, \cdots, L_r].$$

<u>Pullback operations</u>: For an independent square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y,
\end{tikzcd}$$

the pullback operation

$$g^* : \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X' \xrightarrow{f'} Y')$$

is defined by

$$g^*\left(\sum_V n_V[V \xrightarrow{h_V} X; L_1, \cdots, L_r]\right) := \sum_V n_V[V' \xrightarrow{h_V'} X'; g''^* L_1, \cdots, g''^* L_r],$$

where we consider the following fiber squares:

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h_V'}\downarrow & & \downarrow{\scriptstyle h_V} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

(2) Let $\mathcal{OBT}$ be a class of oriented bivariant theories $\mathbb{OB}$ on the same category $\mathcal{V}$ with a class $\mathcal{C}$ of confined morphisms, a class of independent squares, a class $\mathcal{S}$ of specialized morphisms and a class $\mathcal{L}$ of line bundles. Let $\mathcal{S}$ be nice canonical $\mathbb{OB}$-orientable for any oriented bivariant theory $\mathbb{OB} \in \mathcal{OBT}$. Then, for each oriented bivariant theory $\mathbb{OB} \in \mathcal{OBT}$ there exists a unique oriented Grothendieck transformation

$$\gamma_{\mathbb{OB}} : \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{OB}$$

such that for any $f : X \to Y \in \mathcal{S}$ the homomorphism $\gamma_{\mathbb{OB}} : \mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{OB}(X \xrightarrow{f} Y)$ satisfies the normalization condition that

$$\gamma_{\mathbb{OB}}([X \xrightarrow{\mathrm{id}_X} X; L_1, \cdots, L_r]) = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(\theta_{\mathbb{OB}}(f)).$$

It turns out that the above oriented bivariant theory restricted to a map to a point, $\mathbb{OM}^{\mathcal{C}}_{\mathcal{S}}(X \to pt)$, gives rise to what Levine and Morel calls *an oriented Borel–Moore functor with products*. Levine–Morel's algebraic cobordism requires further three more geometric conditions: (1) *Dimension axiom*, (2) *Section axiom* and (3) *Formal Group Law axiom*. Indeed, they call algebraic cobordism *a universal oriented Borel–Moore functor with products of geometric type* [LM3]. We consider similar axioms on the above oriented bivariant theory to get *a bivariant algebraic cobordism*, which specializes to Levine–Morel's algebraic cobordism in the case of a map to a point. A new aspect of this bivariant algebraic cobordism is that its associated contravariant functor is a "cohomological" counterpart of Levine–Morel's algebraic cobordism.

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