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RELATIONS IN THE TAUTOLOGICAL RING OF $\overline{M}_{g,n}$

FUMITOSHI SATO

ABSTRACT. We give a way to produce infinitely many relations in the tautological ring of $\overline{M}_{g,n}$. We will pick some of them to produce the very first family of completely recursive relations in the tautological ring of $\overline{M}_{g,1}$.

1. INTRODUCTION

Let $M_{g,n}$ be the moduli of genus g smooth curves with n distinct marked points defined over the complex numbers. There is a compactification of $M_{g,n}$ denoted by $\overline{M}_{g,n}$ which is the moduli of genus g stable curves with n marked points. A genus g stable curve with n marked points is an arithmetic genus g complete connected nodal curve with distinct smooth n marked points and finite automorphisms. $\overline{M}_{g,n}$ has a stratification according to topological types.

Let $A^*(\overline{M}_{g,n})$ be the Chow ring with $\mathbb{Q}$-coefficients. The system of tautological rings is defined to be the set of smallest $\mathbb{Q}$-subalgebras containing $1(\neq 0)$ of the Chow rings,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n})$$

satisfying the following two properties:

(1) The system is closed under pushforward and pullback via all the maps forgetting the last marking (Figure 1):

$$\pi : \overline{M}_{g,n} \to \overline{M}_{g,n-1}.$$

Figure 1. *Forgetting the last marked point* $*$

(2) The system is closed under pushforward and pullback via all the gluing maps (Figures 2–3):

$$\iota : \overline{M}_{g_1,n_1\cup *} \times \overline{M}_{g_2,n_2\cup \#} \to \overline{M}_{g_1+g_2,n_1+n_2}$$

$$\iota : \overline{M}_{g,n\cup\{*,\#\}} \to \overline{M}_{g+1,n}$$

with attachments along the marking $*$ and $\#$.

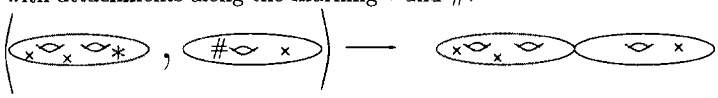


Figure 2. *Gluing* $*$ *and* $\#$

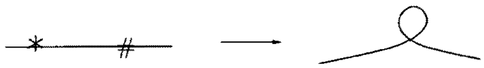


Figure 3. *Gluing* $*$ *and* $\#$

We can define ψ, λ classes on $\overline{M}_{g,n}$ as follows. Consider the forgetting map π , that is forgetting the last marked point,

$$\begin{tikzcd} \overline{M}_{g,n+1} \arrow[d, bend left, "\pi"] \\ \overline{M}_{g,n}. \arrow[u, bend left, "s_i"] \end{tikzcd}$$

$\overline{M}_{g,n+1}$ is the universal curve of $\overline{M}_{g,n}$. So there are n-sections s_i. ψ and λ-classes are defined as follows

$$\lambda_i := c_i(\pi_*(\omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}})),$$
$$\psi_i := c_1(s_i^*(\omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}})).$$

Remark 1. *ψ_i can be defined on $\overline{M}_{g,n}$ for any n. But this notation does not indicate where it lives. They satisfy $\psi_i = \pi^*\psi_i + D_{i,n+1}$ where $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ and $D_{i,n+1}$ is a divisor such that i-th and $n+1$-th points are on the same rational component. on the hand, λ-classes were pullback to λ-classes.*

While the definition of the tautological ring appears restrictive, ψ, κ and λ-classes all lie in the tautological ring [13]. For example,

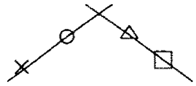
$$-\pi_*(\iota_*([\overline{M}_{g,n}] \times [\overline{M}_{0,3}])^2) = \psi_i$$

where $\iota : \overline{M}_{g,\{1,2,\cdots,i-1,*,i+1,\cdots n\}} \times \overline{M}_{0,\{\#,i,n+1\}} \to \overline{M}_{g,n+1}$.

The tautological rings possess a rich conjectural structure [5].

Example 2. Here are some examples of relation in the tautological ring.

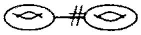
On $\overline{M}_{0,4}$,

$\psi_i =$  .

On $\overline{M}_{1,1}$,

$$\psi_1 - \lambda_1 = 0,$$

$$\lambda = \frac{1}{24}\delta_0.$$

On $\overline{M}_{2,1}$, $\psi_1^2 - \lambda_1\psi_1 + \lambda_2 =$  .

Theorem 3. *(Mumford [13]) In $A^*(M_{g,1})$*

$$\psi_1^g - \lambda_1\psi_1^{g-1} + \cdots + (-1)^g\lambda_g = 0.$$

Theorem 4. *(Graber and Vakil [10]) Every codimension $g+i$ tautological class of $\overline{M}_{g,n}$ is supported on the boundary strata with at least $i+1$ genus 0 components.*

Remark 5. *$R^*(\overline{M}_{g,n})$ is finitely generated , so it is not so surprising to find a relation. We need to find some interesting ones.*

Classically, the tautological ring of $\overline{M}_{g,n}$ was studied by using Riemann–Roch theorem and knowledge of the intersection ring of the moduli of genus g polarized Abelian varieties [13], [7]. But the moduli of curves and the moduli of Abelian varieties are not isomorphic after genus 2. It is very hard to study the tautological ring of $\overline{M}_{g,n}$ for $g > 2$. Thus we need some other method to produce relations in the tautological ring of moduli of higher genus curves. We consider intersections of $\overline{M}_{g,n}$ as the 0-dimensional Gromov–Witten theory. Then there is a principal to do it.

Speculation 6. *(Pandharipande and Y.P. Lee) Any theorem of Gromov–Witten theory should be proved by the localization theory.*

Actually, our motivation of study of the moduli of curves comes from the Gromov–Witten theory.

Consider a forgetting map

$$\begin{array}{c} \overline{M}_{g,n+1}(X,\beta) \\ \downarrow \pi \\ \overline{M}_{g,n}. \end{array}$$

If we have a relation in the tautological ring of $\overline{M}_{g,n}$, then we can pullback the relation to $\overline{M}_{g,n}(X,\beta)$. From it, we can have a recursive formula of Gromov–Witten invariants.

2. Localization

2.1. **Equivariant cohomology.** In this section, we will review equivariant cohomology. Let X be a smooth projective variety with $\mathbb{C}^*$-action. Then we can define $\mathbb{C}^*$-equivariant cohomology group by

$$H^*_{\mathbb{C}^*}(X,\mathbb{Q}) := H^*(X \times_{\mathbb{C}^*} \mathbb{E}\mathbb{C}^*, \mathbb{Q}).$$

where $\mathbb{E}\mathbb{C}^*$ is the universal $\mathbb{C}^*$ principal bundle.

There are several properties which equivariant cohomology will satisfy.

(1) if X =point, $H^*_{\mathbb{C}^*}(X) = \mathbb{Q}[t]$.
(2) If $\mathbb{C}^*$-action is trivial, $H^*_{\mathbb{C}^*}(X) = H^* \otimes \mathbb{Q}[t]$.
(3) If $X/\mathbb{C}^*$ has only finite stabilizers, $H^*_{\mathbb{C}^*}(X) = H^*(X/\mathbb{C}^*)$.
(4) All the functorial properties will be satisfied. Especially $H^*_{\mathbb{C}^*}(X)$ is a $H^*_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{Q}[t]$-module.
(5) If X is projective, $H^*_{\mathbb{C}^*}(X)|_{t=0} = H^*(X)$.

Theorem 7 (The Localization Theorem). *Let $\coprod F_i$ be the fixed loci of X by $\mathbb{C}^*$-action. Then*

$$\oplus(H^*_{\mathbb{C}^*}(F_i) \otimes \mathbb{Q}(t)) \rightleftarrows H^*_{\mathbb{C}^*}(X) \otimes \mathbb{Q}(t)$$

is isomorphism, where $\rightarrow$ is pushforward by inclusion and $\leftarrow$ is pullback by inclusion and then divided by the Euler class.

Remark 8. *The case of Chow group can be done similar way, but in that case, the approximation of $\mathbb{E}\mathbb{C}^* \to \mathbb{P}^\infty$ is needed to define equivariant Chow group. The localization theorem for cohomology was proved by the Leray spectral sequence. On the other hand, the localization theorem in Chow case was proved by using the exact sequence of higher Chow group* [4].

2.2. **Virtual localization.** The higher genus Kontsevich–Manin spaces $\overline{M}_{g,n}(\mathbb{P}^m, d)$ are in general non-reduced, reducible, singular, so we can not apply the usual localization formula [1]. The answer to overcome this difficulty is the virtual localization theorem by Graber and Pandharipande [10].

Theorem 9 (The virtual localization theorem). *(*[10] *§1) Suppose $f : X \to X'$ is a $T = (C^*)^{m+1}$-equivariant map of proper Deligne–Mumford quotient stacks with a T-equivariant perfect obstruction theory. If $i' : F' \hookrightarrow X'$is a fixed substack and $c \in A^*_T(X)$, let $f_{F_i} : F_i \to F'$ be the restriction of f to each of the fixed substacks $F_i \subset f^{-1}(F')$. Then*

$$\sum_{F_i} f_{F_i *} \frac{i^*_{F_i} c}{\epsilon_T(F_i^{vir})} = \frac{i'^* f_* c}{\epsilon_T(F'^{vir})}$$

where $i_{F_i} : F_i \hookrightarrow X$ and $\epsilon_T(F^{vir})$ is the virtual equivariant Euler class of "virtual" normal bundle F^{vir}.

Remark 10.
(1) *If X and X' are nonsingular with the trivial perfect obstruction theories (*[3] *§4), then the virtual localization formula reduces to the standard localization formula.*
(2) *The conditions in the theorem are satisfied for the Kontsevich–Manin spaces $\overline{M}_{g,n}(\mathbb{P}^m, d)$ with the induced action by the diagonal action of T on $\mathbb{P}^m$, and $\epsilon_T(F^{vir})$ can be explicitly computed in terms of ψ and λ-classes (*[10] *§4).*

2.3. **$\mathbb{C}^*$-action on $\mathbb{P}^1$.** We define a $T = \mathbb{C}^*$-action on $\mathbb{P}^1$ for $a \in T$ and $(x_0 : x_1) \in \mathbb{P}^1$ by $a \cdot (x_0 : x_1) = (x_0 : ax_1)$. There are two fixed points $0 = (0 : 1)$ and $\infty = (1 : 0)$.

This T-action induces T-actions on $\overline{M}_{g,n}(\mathbb{P}^1, d)$.

3. Producing relations in $R^*(\overline{M}_{g,n})$

In this section, we will produce infinitely many relations in $R^*(\overline{M}_{g,n})$ by using the localization theory which was explained in the previous section.

Consider the following map,

$$f : \overline{M}_{g,m}(\mathbb{P}^1, d) \to \overline{M}_{g,m} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

We choose a fixed loci $\overline{M}_{g,m} \times 0 \times \cdots \times 0 \times \infty \times \cdots \times \infty$. If we take $c = [1]^{\text{vir}}$, we can obtain a relation on $\overline{M}_{g,m}$.

To obtain a relation on $\overline{M}_{g,m}$, we can pushforward the relation by π's. The other thing which we can do is to multiply by ψ in the each step. It seems that multiplying by ψ is not that important, but this process will kill contribution of many fixed loci which you don't like to show up in the end, because $\overline{M}_{0,4}$ is 0-dimensional. The other important point is that one can compute the pushforward explicitly by using the dilaton equation. So far we don't know in general how to compute the pushforward explicitly in term of λ and ψ-classes if we multiply higher power of ψ and then pushforward.

Remark 11. *We did not use κ-classes at all in this paper. The main reason is that λ-classes are natural classes which appear in numerators in the "virtual" localization theorem. Maybe we can have interesting relations by multiplying by higher power of ψ-class if we use κ-classes instead of λ-classes.*

Let us give one example of interesting relations which were obtained by the above method [2].

We will apply the above method twice. The first map is $f : \overline{M}_{g,2}(\mathbb{P}^1, 1) \to \overline{M}_{g,2} \times \mathbb{P}^1 \times \mathbb{P}^1$ and take its fixed locus is $\overline{M}_{g,2} \times 0 \times \infty$. The second map is $f : \overline{M}_{g,1}(\mathbb{P}^1, 1) \to \overline{M}_{g,1} \times \mathbb{P}^1$ and its fixed locus is $\overline{M}_{g,1} \times 0$.

Theorem 12. *In the tautological ring of $\overline{M}_{g,1}$,*

$$\Sigma_{i=0}^{g}(-1)\lambda_i\psi_1^{g-i} = \Sigma_{h=1}^{g-1}(1 - \frac{h}{g})\iota_{h*}c_h,$$

where $\iota_h : \overline{M}_{h,2} \times \overline{M}_{g-h,1} \to \overline{M}_{g,1}$ and

$$c_h := \Sigma_{i=0}^{g-1}(-1)^{h+i}(\Sigma_{j=0}^{h}(-1)^j\lambda_j^0\psi_0^{i-j})(\Sigma_{j=0}^{g-h}\lambda_j^\infty\psi_\infty^{g-1-i-j}).$$

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SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, KOREA
E-mail address: `fumi@kias.re.kr`