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Dual varieties, ramification, and Betti numbers of projective varieties

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Introduction

A nondegenerate nonsingular complex projective algebraic variety $X^n \subset \mathbb{P}^N$ has many well understood numerical invariants, such as dimension n, codimension $a = N - n$, degree d, classes μ_i (cf. below) etc. However, except for the dimension, these invariants depend on embedding and do not characterize X as an abstract variety. Viewed as an abstract variety, X has other numerical invariants, such as products of Chern classes (e.g. the selfintersection (K_X^n), where K_X is the canonical class of X). However, the most important numerical invariants of the abstract variety X are probably its Hodge numbers $h^{p,q}(X)$ and Betti numbers $b_i(X) = \sum_{q=0}^{i} h^{i-q,q}(X)$. Thus it is reasonable to ask what are the possible values of "abstract" numerical invariants for a given projective variety X and in particular what are the restrictions on topology imposed by being contained in a given projective space with given invariants?

The case $n = 1$, $N = 2$ is classical and easy. If g is the genus of X and $b_1 = 2g$ is the first Betti number, then $b_1 = (d-1)(d-2)$, so that the "abstract" invariants are determined by the projective ones (this is no surprise since it is clear that all nonsingular plane curves are diffeomorphic to each other). These important formulas which seem obvious to us are actually quite deep (especially the relationship between the genus and the first Betti number) and are due to Riemann.

The situation is similar in the case when $N = n+1$, i.e. for hypersurfaces of arbitrary dimension (for the Betti numbers one can use the same argument while the claim for the Hodge numbers follows from the one for the Betti numbers and semicontinuity). But already the case of space curves is much more complicated. For example, there exist two types of nonsingular curves of degree four on a nonsingular quadric in $\mathbb{P}^3$, viz. a rational curve of type $(3,1)$ and an elliptic one of type $(2,2)$. The case of space curves was settled by Halphen in the paper [Hal] bringing him (together with Noether) the Steiner prize from the Berlin Academy of Sciences. Roughly speaking, Halphen proved that if $X \subset \mathbb{P}^3$ is a curve of degree d and m is the minimum of the degrees of surfaces containing X, then $b_1(X)$ does not exceed $\frac{d^2}{m} + \cdots$, where dots stand for a linear function in d, and, under certain assumptions, described the curves of maximal genus for given d and m. In particular, from Halphen's results it follows that the first Betti number of a nonsingular space curve does not exceed $\frac{d^2}{2} + \cdots$, i.e. is, roughly, at least twice less than that of plane curves of the same degree, and the curves with maximal Betti number lie on a quadric. A modern treatment and refinement of Halphen's theory was given by Gruson and Peskine [G-P] and Harris [Har].

In the case when $X \subset \mathbb{P}^N$ is a nondegenerate nonsingular curve of arbitrary codimension $a = N-1$, Castelnuovo proved (cf. [Cast1] and [Cast2]) that $b_1(X) \le \frac{d^2}{a} + \cdots$ and classified the curves for which b_1 attains maximal value for given d and a; it turns out that such a curve has to lie on a surface of (minimal) degree $N-1$ in $\mathbb{P}^N$ (an exposition of Castelnuovo's theory can be found in [G-H, Ch. 2, §3 and Ch. 4, §3]; of course, the bounds in Halphen's and Castelnuovo's theories are quite explicit, but for the sake of introduction we are satisfied with their leading terms).

Castelnuovo's beautiful theory was generalized in many directions, but the only generalization to varieties of dimension larger than one I know of is due to Harris [Har2]. To wit, Harris gives a sharp bound for the geometric genus $p_g = h^{n,0}$ of an arbitrary nondegenerate variety $X \subset \mathbb{P}^N$ and classifies the varieties on the boundary. Harris' s result is as powerful as Castelnuovo's one and is deduced from it by induction. However, the geometric genus for higher dimensional varieties does not play a role comparable to that of genus of curves. Therefore it is desirable to prove Castelnuovo type theorems for other Hodge and Betti numbers of varieties of arbitrary dimension.

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Castelnuovo bound for curves can be interpreted in several different ways. We presented it as a bound for the first Betti number b_1, but one may also think of it as a bound for r_1, where r_1 is the degree of the ramification divisor of a general projection of X onto $\mathbb{P}^1$ (by the Riemann-Hurwitz formula, $r_1 = 2g + 2d - 2$). Alternatively, one can think of Castelnuovo's bound as a bound for d^*, where $d^* = \mu_1$ is the class of the curve X, i.e. the degree of the dual hypersurface $X^* \subset \mathbb{P}^{N^*}$ (it is immediate that $d^* = r_1$).

Castelnuovo bounds for r_1 and μ_1 generalize to the higher dimensional case, but there they yield different bounds. Comparing them and using bounds for Betti numbers in terms of classes, we will finally get a bound for Betti numbers in terms of dimension, codimension, and degree.

Bounding Betti numbers of algebraic varieties has been a popular pursuit for the last century and a half, and here we only mention some landmarks important for the development of this topic. We already described bounds for the genus of complex projective curves. Most subsequent papers on Betti numbers dealt primarily with real varieties; results for complex varieties were derived from these by forgetting the complex structure, which, of course, resulted in loss of information. First general results on topology of real algebraic varieties were obtained by Harnack [Harn] who proved that the number of ovals (i.e. the first Betti number b_1 which in this setup equals b_0) of a nonsingular real plane algebraic curve of degree d does not exceed $\frac{(d-1)(d-2)}{2} + 1$. Considerable progress in the study of Betti numbers was made by Petrovskiĭ and Oleĭnik [Ol] who, however, like Harnack, considered only the case of hypersurfaces. Milnor [Mil2] and Thom [Th] independently and almost simultaneously obtained bounds for the total Betti number $b = \sum_i b_i$ valid for algebraic varieties of arbitrary codimension, both real and complex ones. Milnor "deforms" an arbitrary real affine variety V which can be defined by equations of degree $\leq k$ to a smooth affine hypersurface of degree $\leq 2k$ and uses an Oleĭnik type bound for hypersurfaces to give a bound for the total Betti number of V in terms of k and the dimension of the ambient affine space. Bounds for complex algebraic varieties are obtained as a consequence of the bounds in the real affine case; to obtain them Milnor considers separately the real and imaginary parts of coordinates and equations, which results in further loss of sharpness. The projective case is reduced to the affine one. Using the inequality $k \leq d$ (and thus passing from the degrees of equations to the degree of variety), one can rewrite Milnor-Thom's inequality for real projective varieties in the form $b(X_{\mathbb{R}}) < c \cdot d^{N+1}$ and for complex projective varieties in the form $b(X_{\mathbb{C}}) < c \cdot d^{2N+2}$, where $c = c(N)$ is a constant depending on the dimension of the ambient $\mathbb{P}^N = \langle X \rangle$. As Milnor himself pointed out, these estimates are "presumably rather crude" and "certainly not best possible". Recently Laszlo and Viterbo [L-V] combined some bounds for Chern classes of complex manifolds with Smith theory to obtain a bound of the form $b < c \cdot d^{n+1}$, where $n = \dim X$ and $c = c(n)$, valid both for complex and real projective varieties (in the latest case only for homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$). Unfortunately, their bound also fails to be sharp, and so there is no question of describing varieties on the boundary. The motivation behind [Mil2], [L-V] and other papers on a similar topic was that total Betti number measures "complexity" of variety, so that bounding this number and exploring varieties on the boundary will give a bound of complexity and an idea of what the "most complex" varieties look like. However, as we will see, varieties with big Betti number (the so called *Castelnuovo varieties* are actually very simple from the point of view of algebraic geometry.

In this paper we give sharp (or asymptotically sharp) bounds for the total Betti number (with arbitrary coefficients) of nonsingular complex projective varieties and explain where to look for varieties on the boundary. These bounds generalize Castelnuovo bound for curves and strengthen the bounds for the total Betti number mentioned above. We also get bounds for the individual Betti numbers, but these are sharp (or asymptotically sharp) only in the case of middle homology.

As we already pointed out, we derive bounds for Betti numbers from bounds for classes of complex projective varieties (cf. below). This can be done in two ways, viz. using Lefschetz theory or adapting Morse theory to study projective varieties. The first method works over an arbitrary algebraically closed field, but yields somewhat weaker results. Here we use the second method. While classical Morse theory deals with compact manifolds, here we consider its relative version over a projective line. This allows us to study Lefschetz pencils and to develop what we call *Morse-Lefschetz theory*.

In the first section we show how the classical invariants called *classes* can be used to bound or even compute important topological invariants of nonsingular affine or projective algebraic varieties. If $X \subset \mathbb{P}^N$ is an n-dimensional projective variety, then the n-th class or simply the *class* $\mu_n = \mu_n(X)$ is defined as the degree d^* of the dual variety $X^* \subset \mathbb{P}^{N^*}$ (called *codegree*) provided that X^* is a hypersurface in P^{N^*} and zero otherwise (we recall that X^* is the locus of tangent hyperplanes to X, i.e. the hyperplanes containing the projective tangent space $T_{X,x}$ at a nonsingular point $x \in X$). For the purpose of this introduction (and even the paper as a whole), one can define the *i-th class* $\mu_i(X)$, $0 \leq i \leq n$ as the class of the intersection of X with a general linear subspace of codimension $n-i$ in $\mathbb{P}^N$. In particular, μ_0 is equal to the degree d of X and μ_1 is the class of a general curve section C of X

which by the Riemann-Hurwitz formula equals $2g_C + 2d - 2$. It turns out that when X is nonsingular one has $b_i = b_{2n-i} \le \mu_i + \mu_{i-2} + \mu_{i-4} + \dots$, $i \le n$ and $e = (n+1)\mu_0 - n\mu_1 + (n-1)\mu_2 - \cdots$, where e is the Euler-Poincaré characteristic of X. We get even more simple bounds and formulas for the Betti numbers and Euler characteristic of a nonsingular affine variety in terms of the classes of its projective closure. Our results give a better understanding of and strengthen Lefschetz's theorems.

The results of the first section reduce the problem of bounding the Betti numbers of a nonsingular projective variety X to that of bounding the classes of X. Classes and, particularly, the codegree are themselves important invariants of projective varieties. For example, while varieties of codegree one and two are, respectively, linear spaces and quadrics, classification of varieties of codegree three is already deep and nontrivial (cf. [Za, Ch. IV, Theorem 5.2]) and classification of varieties of codegree four has not yet been completed. In [Za2] we prove various sharp lower bounds for d^* and classify the varieties on the boundary. However, to bound the Betti numbers from above we need *upper* bounds for classes. In the second section we give upper bounds for classes in terms of degrees of selfintersections of ramification divisors. The ramification divisor $R \subset X$ of a general linear projection $p\colon X \to \mathbb{P}^n$ is a handy tool to explore the geometry of X, often more convenient than the canonical class K (one has $R \sim K + (n+1)H$, where H is a hyperplane section) because it is ample [Za, Chapter I, Corollary 2.14] and even very ample [Ein]. Put $r_i = \deg R^i = (R^i H^{n-i})$. Using the Hodge index theorem, we show that the subsequent quotients $\frac{r_i}{r_{i-1}}$ form a nonincreasing sequence, i.e. $\frac{r_1}{r_0} \ge \frac{r_2}{r_1} \ge \cdots \ge \frac{r_n}{r_{n-1}}$, and so $r_i \le \frac{r_1^i}{r_0^{i-1}} = \frac{r_1^i}{d^{i-1}}$. Clearly, for $n > 1$ the number r_1 is stable under passing to a general hyperplane section, and thus is bounded by Castelnuovo's theorem for curves. On the other hand, we show that the i-th class is bounded by r_i, viz. $\mu_i \le r_i$. Thus the results of section 2 yield an upper bound for the classes in terms of dimension, codimension and degree. We also obtain a universal sharp bound for classes in terms of dimension and degree; this bound does not involve codimension. Finally, in section 2 we get a sharp bound for the classes of varieties of given dimension n and codimension a defined by equations whose degree does not exceed d.

Combining the bounds for classes obtained in section 2 with the bounds for Betti numbers from section 1, in section 3 we get bounds for Betti numbers in terms of dimension n, codimension a, and degree d. The easiest application is that hypersurfaces have the maximal Betti number among all varieties of given dimension and degree. We also obtain a bound for Betti numbers of varieties of given dimension n and codimension a defined by equations whose degree does not exceed d.

In section 4 we explain where to look for varieties with maximal invariants (be it ramification degree, class or Betti numbers) and obtain sharp bounds for these invariants (at least for varieties of sufficiently large degree). Furthermore, we discuss some generalizations (of special importance is the extension of our results to Chern and Hodge numbers) and open problems.

While giving precise statements and explaining our approach from different points of view (particularly in the first three chapters), in this paper we do not give proofs; detailed proofs will appear elsewhere.

Notations and conventions

Throughout the paper we consider nondegenerate varieties, that is, varieties spanning the ambient linear space. Thus X denotes a nondegenerate n-dimensional projective variety in $\mathbb{P}^N$, $L_\infty \subset \mathbb{P}^N$ the hyperplane "at infinity", $X_\infty = L_\infty \cap X$. For a nondegenerate variety $X^n \subset \mathbb{P}^N$, we denote by a its codimension: $a = \operatorname{codim} X = N - n$. We denote by X_i, $0 \le i \le n$ the section of X by a general linear subspace $L_i \subset \mathbb{P}^N$, $\operatorname{codim} L_i = n - i$, so that X_i is a nondegenerate projective variety of dimension i. Sometimes we also denote X_{n-1} by X' and X_{n-2} by X''. The affine variety $X \setminus X_\infty$ will be usually denoted by V. For a variety Y we denote by $\operatorname{Sm} Y$ (resp. $\operatorname{Sing} Y$ the set of smooth (resp. singular) points of Y. For a real variety X (resp. V), the set of its real points will be sometimes denoted by $X_{\mathbb{R}}$ (resp. $V_{\mathbb{R}}$ and the set of complex points by $X_{\mathbb{C}}$ (resp. $V_{\mathbb{C}}$).

We use standard notations and conventions for homology, relative homology, and cohomology. As for Betti numbers, we usually state our results for $b_i(M) = b_i(M, \mathbb{F}) = \dim H_i(M, \mathbb{F})$, where $\mathbb{F}$ is a field. However, since we deduce statements for Betti numbers from those for cell complexes, usually our bounds are actually stronger. To wit, let $H_i(M, \mathbb{Z}) = \mathbb{Z}^{\beta_0} \oplus (\mathbb{Z}/m_1\mathbb{Z})^{\beta_1} \oplus \cdots \oplus (\mathbb{Z}/m_t\mathbb{Z})^{\beta_t}$, where $m_1, \dots, m_t$ are distinct powers of prime numbers, be a primary decomposition. Then our bounds are usually valid if we put $b_i = \sum_{j=0}^t \beta_j$. We denote by $b = b(M) = \sum_i b_i$ the *total* Betti number and by $e = e(M) = \sum_i (-1)^i b_i(M, \mathbb{Q})$ the *Euler-Poincaré characteristic* of M.

1. Morse-Lefschetz theory for complex algebraic varieties

Lefschetz's book [Lef] on the topology of complex projective varieties appeared even earlier than the paper in which Morse laid foundations of what is now called Morse theory. One of the main issues dealt with in this book was to compare the topology of a nonsingular complex projective variety with that of its general hyperplane section (cf. [Lam] for a modern exposition of the Lefschetz theory). In particular, Lefschetz expressed the class of variety in terms of Betti numbers of its linear sections (cf. [Lef, Ch. V, Th. XII] or [Lam, (3.5.3)] (although usually attributed to Lefschetz, this was actually done by Alexander in [Al] in terms of the multidimensional Zeuthen-Segre invariant introduced by Segre in [Seg]; cf. Historical Remark 1.12 (i) below). We go the other way round and give bounds for the Betti numbers in terms of classes of (linear sections of) the variety. These bounds can be obtained by developing Lefschetz (or Alexander) type arguments. However, we prefer using a refinement of Morse theory which yields more detailed information and applies to homology with arbitrary coefficients. The idea of using Morse theory to prove Lefschetz theorems is by no means new (it was probably first suggested by Thom and realized by Andreotti and Frankel and Bott), but using a relative version of Morse theory we will be able to link in a very natural way topological properties of varieties to classical projective invariants defined in terms of dual varieties. The point is that while Andreotti and Frankel and others applied Morse theory to the function defined by the distance to a fixed point of the ambient space (which does not seem very natural in the projective setting) we follow Lefschetz in using the good old method of fibering our variety by means of a general pencil of hyperplane sections. That explains our usage of the term "Morse-Lefschetz Theory". We will not give details of our refinement of Morse theory, but rather discuss some of its applications in the context of complex algebraic varieties.

We start with recalling some definitions. Let $X \subset \mathbb{P}^N$ be an n-dimensional complex projective variety. The hyperplanes in $\mathbb{P}^N$ are parametrized by the dual projective space $\mathbb{P}^{N^*}$. For a point $\alpha \in \mathbb{P}^{N^*}$ we denote by $L_\alpha \subset \mathbb{P}^N$ the corresponding hyperplane in $\mathbb{P}^N$. For a point $x \in X$, we denote by $T_{X,x} \subset \mathbb{P}^N$ the (embedded projective) tangent subspace to X at x; if $x \in \operatorname{Sm} X$, then $\dim T_{X,x} = \dim X = n$. A hyperplane $\alpha \in \mathbb{P}^{N^*}$ is said to be *tangent* to X at x if $L_\alpha \supset T_{X,x}$.

Definition 1.1. The subvariety

$$\mathcal{P}_X \subset X \times \mathbb{P}^{N^*}, \qquad \mathcal{P}_X = \overline{\{(x,\alpha) \mid x \in \operatorname{Sm} X, L_\alpha \supset T_{X,x}\}},$$

where bar denotes projective closure, is called the *conormal variety* of X in $\mathbb{P}^N$, and its image X^* under the projection of $\mathbb{P}^N \times \mathbb{P}^{N^*}$ onto the second factor is called the *dual variety* of X.

Thus X^* is the locus of hyperplanes that are tangent to X at a nonsingular point. Furthermore, since the fiber of $\mathcal{P}_X$ over $x \in \operatorname{Sm} X$ is the $(N-n-1)$-dimensional linear subspace of $\mathbb{P}^{N^*}$ dual to $T_{X,x}$, one has $n^* = \dim X^* \le \dim \mathcal{P}_X = N-1$. The number $\operatorname{def} X = N - n^* - 1$ is called the *defect* of X; if $\operatorname{def} X > 0$, then X is called *defective*.

Definition 1.2. Denote by $d = \deg X$ the degree of X and by $d^* = \operatorname{codeg} X = \deg X^*$ the degree of X^*, i.e. the number of common points of X^* and a general $(\operatorname{def} X + 1)$-dimensional linear subspace in $\mathbb{P}^{N^*}$. Then d^* is called the *codegree* and $\mu = \mu_n = \begin{cases} d^*, & \operatorname{def} X = 0, \\ 0, & \operatorname{def} X > 0 \end{cases}$ the *class* of X.

Let $0 \le i \le n$, and let X_i be a general i-dimensional linear section of X. The class of X_i is called the *i-th class* of X and is denoted by μ_i (the i-th class is also called the *$(n-i)$-th rank* of X). Thus $\mu_n = \mu$ is the class of X, $\mu_0 = d = \deg X$ and $\mu_i = 0$ iff $\operatorname{def} X_i = 0$, i.e. iff $i > n - \operatorname{def} X$.

The number $\boldsymbol{\mu}_i = \sum_{j=0}^{i} \mu_j$, is called the ***i-th cumulative class*** (resp. the ***i-th reduced cumulative class***) of X, and the number $\boldsymbol{\mu} = \boldsymbol{\mu}_n = \sum_{j=0}^{n} \mu_j = \sum_{j=0}^{n-\operatorname{def} X} \mu_j$ the ***total class*** of X.

It is clear that $\mu_i(X) = \mu_i(X_{n-1}) = \cdots = \mu_i(X_i)$, and in particular $\boldsymbol{\mu}(X) = \mu_n(X) + \boldsymbol{\mu}(X_{n-1})$. Furthermore, μ_i can be interpreted as the degree of the i-dimensional polar locus $P_i = P_i(L) = \{x \in X \mid \dim T_{X,x} \cap L \ge i-1\}$, where $L \subset \mathbb{P}^N$ is a general linear subspace of codimension $n-i+2$, and can be computed as an intersection on the conormal variety: $\mu_i = \int_{\mathcal{P}_X} h^{n-i} h'^{N-n+i-1}$, where h and h' denote the liftings on $\mathcal{P}_X$ of the classes of hyperplane sections of X and X^* respectively.

The notions of dual variety and classes were introduced and studied by Poncelet and Plücker in the e of plane curves and by Cayley, Salmon and Zeuthen in the case of surfaces. We refer to [Tev] for position of the theory of dual varieties in characteristic zero. Foundations of a general theory of arieties and classes were laid by Severi and developed by Todd, cf. [Pi] or [Kl].

llowing results strengthen Lefschetz's theorems (cf. [Mil, Theorems 7.1, 7.2 and 7.3]). They seen as striking counterparts of the main theorem of Morse theory (cf. [Mil, Theorem 5.2]), where compact manifold M is replaced by nonsingular complex affine variety $V = X \setminus X_\infty$ and the critical points of index i of a Morse function on M correspond to the singular points of intersections of $V_i = X_i \setminus X_\infty$ with the hyperplanes from a general pencil containing L_∞.

Theorem 1.3. *Suppose that V is smooth. Then V has homotopy type of a cell complex with at most $\mu_i = \mu_i(X)$ cells of dimension i, $i = 0, \dots, n$. Thus $b_i(V) \le \mu_i$, $i = 0, \dots, n$ (so that, in particular, $b_i(V) = 0$ for $i > n - \operatorname{def} X$) and $b(V) \le \mu(X)$.*

If, moreover, both X and X_∞ are smooth, then V has homotopy type of a cell complex with exactly $\mu_i = \mu_i(X)$ cells of dimension i, $i = 0, \dots, n$ and $e(V) = \sum_{i=0}^{n} (-1)^i \mu_i(X)$.

Remark **1.4.**

(i) The bound in Theorem 1.3 is not sharp. For example, since V is connected, $b_0(V) = 1 < \mu_0$, and so $b_1(V) \le \mu_1 - \mu_0 + 1$ and $b(V) \le \mu - 2\mu_0 + 2$.

(ii) As was pointed out by Lê Dũng Tráng, the formula for the Euler-Poincaré characteristic in the second part of Theorem 1.3 can be deduced from the formulas for the local Euler obstructions obtained in [L-T].

Theorem 1.5. *Let $V = X \setminus X_\infty$ be a nonsingular n-dimensional affine variety, and let $j\colon X_\infty \hookrightarrow X$ be the natural inclusion. Let $j_i\colon H_i(X_\infty, \mathbb{Z}) \to H_i(X), \mathbb{Z}$ be the corresponding homology map. Then*

(i) *For $i < n - 1$, j_i is an isomorphism;*

(ii) *j_{n-1} is an epimorphism with $\dim(\operatorname{Ker} j_{n-1}) \le \mu_n - \nu_n \le \mu_n$.*

Suppose furthermore that X and X_∞ are nonsingular, and let

If furthermore $\mathbb{F} = \mathbb{C}$ (or, more generally, $\mathbb{F}$ is a field of characteristic zero) and let $j_i^{\mathbb{F}}\colon H_i(X_\infty, \mathbb{F}) \to H_i(X), \mathbb{F}$, where $\mathbb{F}$ is a field of characteristic zero. Then

(iii) *For $i \ge n$, $j_i^{\mathbb{F}}$ is a monomorphism with $\dim(\operatorname{Coker} j_i^{\mathbb{F}}) \le \mu_{2n-i}$.*

Corollary 1.6. *Let $V = X \setminus X_\infty$ be a nonsingular affine n-dimensional variety. Then*

$$\begin{aligned} b_i(X) &= b_i(X_\infty) && \text{for } i < n-1,\\ b_{n-1}(X_\infty) - \mu_n \le b_{n-1}(X) &\le b_{n-1}(X_\infty), &&\\ b_i(X_\infty) - \mu_{2n-i-1} \le b_i(X) &\le b_i(X_\infty) + \mu_{2n-i}(X) && \text{for } i \ge n. \end{aligned}$$

Remarks **1.7.**

(i) The nonsingularity assumption is essential for the validity of Claim (iii) in Theorem 1.5. For example, let $X = v_m(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+m}{n}-1}$ be a Veronese variety, and let X_∞ be its reducible hyperplane section (e.g. a section corresponding to a union of m hyperplanes in $\mathbb{P}^n$). Then $b_{2n-2}(X_\infty) > 1$ (e.g $= m$) while $b_{2n-2}(X) = b_{2n-2}(\mathbb{P}^n) = 1$.

(ii) The bounds in Corollary 1.6 are not sharp. For example, it is easy to show that $\dim(\operatorname{Ker} j_{n-1}) \le \frac{\mu_n}{2}$ (cf. e.g. [Za3, Remark 2.10 (ii)]. On the other hand, Remark 1.4 shows that the last upper bound in Corollary 1.6 fails to be sharp by $\mu_0 - 1$ for $i = 2n-1, 2n$ (furthermore, by the Riemann-Hurwitz formula, $b_{2n-1}(X) \le b_1(X_1) = \mu_1 - 2(\mu_0 - 1)$).

Corollary 1.8. *In the assumptions of Corollary 1.5 one has:*

$$b(X) \le b(X_\infty) + \mu - 2(\mu_0 - 1) < b(X_\infty) + \mu(X).$$

Corollary 1.9. *Let $X^n \subset \mathbb{P}^N$ be a nonsingular projective variety. Then*

$$\begin{aligned} b(X) \le \sum_{i=0}^{n} (\mu(X_i) - 2(\mu_0 - 1)) &= \sum_{i=1}^{n} (n - i + 1)\mu_i - (n+1)(\mu_0 - 2)\\ &= \sum_{i=1}^{\operatorname{def} X} (n - i + 1)\mu_i - (n+1)(\mu_0 - 2). \end{aligned}$$

Corollary 1.9 is also a consequence of the following

Theorem 1.10. *Let Let $X^n \subset \mathbb{P}^N$ be a nonsingular projective variety. Then*

$$b_i(X) = b_{2n-i}(X) \le \sum_{k=0}^{[\frac{i}{2}]} \mu_{i-2k} - (\mu_0 - 1), \quad i \le n,$$
$$b_1(X) = b_{2n-1}(X) = \mu_1 - 2(\mu_0 - 1),$$
$$e(X) = \sum_{i=0}^{n} (-1)^i (n-i+1)\mu_i = \sum_{i=0}^{n-\operatorname{def} X} (-1)^i (n-i+1)\mu_i.$$

Remark **1.11**. The expression of bounds for Betti numbers in Theorem 1.10 as a sum of shifted classes reflects Lefschetz's primitive decomposition. It would be nice to infer Theorem 1.10 from a projective counterpart of Theorem 1.3. Such a result would strengthen Lefschetz's hard theorem just as Theorem 1.3 strengthens Lefschetz's weak theorem.

Historical Remarks **1.12**.

(i) Given basic nature and simplicity of the formulas in Theorem 1.10, one might expect that they should have been known in classical algebraic geometry. Studying the literature we found that this is indeed the case, albeit implicitly, with regard to the Euler-Poincaré characteristic.

For curves, the Riemann-Hurwitz formula shows that $b_1 = \mu_1 - 2\mu_0 + 2 \le \mu_1$ and $e = 2 - b_1 = 2\mu_0 - \mu_1$ in accordance with Theorem 1.10.

The case of surfaces is more interesting. The role of Euler-Poincaré characteristic e was classically played by the *Zeuthen-Segre invariant* I defined by the formula $I = \delta - \sigma - 4p$, where, for a (general) pencil of curves, δ is the number of nodal curves, σ is the number of base points, and p is the genus of a generic curve (cf. [SR, Ch. IX, §7.1]). This number was shown to be independent of the choice of pencil of curves. By methods anticipating those of modern Lefschetz theory, Alexander proved (in two different ways) the equality $I = e - 4$ (cf. [Al, §2 and §3]. In our setup, i.e. for a Lefschetz pencil, using the formula for the genus of a general hyperplane section given in the preceding paragraph, one has, in our notations, $I = \mu_2 - \mu_0 - 4p = \mu_2 - \mu_0 - 2(\mu_1 - 2\mu_0 + 2) = \mu_2 - 2\mu_1 + 3\mu_0 - 4$. By Alexander's interpretation of I, this classical formula (cf. e.g. formula (5) in [S-R, Ch. IX, §7]) can be rewritten as $e = 3\mu_0 - 2\mu_1 + \mu_2$, which is a special case of Theorem 1.10 for $n = 2$. Furthermore, by Lefschetz's theorem (cf. Corollary 1.5) and the formula for curves in the preceding paragraph, $b_3 = b_1 \le 2p = \mu_1 - 2\mu_0 + 2 \le \mu_1$ and $b_2 = e + 2b_1 - 2 \le \mu_2 - 2\mu_1 + 3\mu_0 + 2(\mu_1 - 2\mu_0 + 2) - 2 = \mu_2 - \mu_0 + 2 \le \mu_2$, which again is a special case of Theorem 1.10. It is amusing to observe that the now obvious formula $b_2 = I + 4(p_g - p_a) + 2$ (where p_g and p_a are respectively the geometric and arithmetic genus of our surface), was obtained only by Alexander [Al] who corrected an erroneous formula $b_2 = I + 2(p_g - p_a) + 2$ due to Poincaré and published in 1906.

In [Seg, §11] C. Segre introduced a generalization II of the invariant I to varieties of arbitrary dimension by using (in our notation) a recurrent formula $II_n = \mu_n - 2II_{n-1} - II_{n-2}$, $II_0 = \mu_0 - 1$, $II_1 = \mu_1 - 2\mu_0 + 2$ (thus $II_2 = I + 1$; normalization chosen by Segre is such that the invariants II_n vanish for all linear spaces). Computing by induction, one can show that

$$\begin{aligned} II_n &= (-1)^n \left(\sum_{i=0}^{n} (n-i+1)\mu_i - n - 1 \right) \\ &= (-1)^n \left(\mu_n - 2\mu_{n-1} + \dots + (-1)^{n-1} n\mu_1 + (-1)^n (n+1)(\mu_0 + 1) \right). \end{aligned}$$

On the other hand, using Alexander's method, one can verify that $II_n = (-1)^n(e - n - 1)$ (Alexander himself used a different generalization I_n of I defined by the same recurrent formula, but basing on a different value $I_0 = II_0 + 1$, which does not make much difference). Putting together the above two formulas, one gets the equality $e(X) = (n+1)\mu_0 - n\mu_1 + \dots + (-1)^n \mu_n$ from Theorem 1.10.

We do not know of any classical bounds for the Betti numbers in terms of classes similar to those in Theorem 1.10.

(ii) One can view the formulas for the Euler-Poincaré characteristic in Theorems 1.3 and 1.10 as generalizations of Hopf's celebrated formula for the number of singularities of a vector field to the setup of projective algebraic geometry.

2. Bounds for projective invariants

Theorem 2.1. *Let $X \subset \mathbb{P}^N$ be a nonsingular projective variety of dimension n, and let R be an ample divisor on X. Denote by H a hyperplane section of X, and let $d = (H^n)$ be the degree of X and $r_i = (R^i H^{n-i})$ the degree of R^i (so that, in particular, $r_0 = d$, $r_1 = \deg R$ and $r_n = r = (R^n)$). Then the subsequent quotients $\frac{r_i}{r_{i-1}}$ form a nonincreasing sequence, i.e. $\frac{r_1}{r_0} \ge \frac{r_2}{r_1} \ge \cdots \ge \frac{r_n}{r_{n-1}}$, and $r_i \le \frac{r_{i-1}^2}{r_{i-2}} \le \cdots \le \frac{r_j^{i-j+1}}{r_{j-1}^{i-j}} \le \cdots \le \frac{r_1^i}{r_0^{i-1}} = \frac{r_1^i}{d^{i-1}}$, $1 \le i \le n$.*

Remarks **2.2.**

(i) Let X_k be the section of X by a general linear subspace of codimension k in $\mathbb{P}^N$, $0 \le k \le n$. Then it is clear that $r_i(X) = r_i(X_k)$, $0 \le i \le n-k$.

(ii) The bounds in Theorem 2.1 are sharp. In fact, it is clear that if $R \sim aH$ for some $a > 0$, then $r_i = a^i d$, $i = 0, \dots, n$ and all the inequalities in Theorem 2.1 turn into equalities.

(iii) The bound in Theorem 2.1 is much better than a general one given in [Fu, Example 8.4.7] (viz. $r_i \le r_1^i d^{n-i}$), but, unlike Fulton's bound, our bound does not hold if we replace i copies of R by i divisors $R_1, \dots, R_i$ that are not equivalent to each other. To see this it suffices to consider the nonsingular quadric $X \subset \mathbb{P}^3$ and two ample divisors $R_1 \sim \ell_1 + 2\ell_2$, $R_2 \sim 2\ell_1 + \ell_2$, $d_i = \deg R_i = 3$, $i = 1, 2$, where ℓ_1 and ℓ_2, $(\ell_1 \cdot \ell_2) = 1$ are two generators. Then $d(R_1 \cdot R_2) = 10 > d_1 d_2 = 9$.

(iv) In the proof of Theorem 2.1 we do not make full use of the assumption that H is a hyperplane section. For example, the theorem stays true if we only assume that $d = (H^n) > 0$ and (some multiple of) H does not have base points.

(v) It seems that the hypothesis that X is nonsingular can be substantially weakened. We deduce Theorem 2.1 from the Hodge index theorem and, replacing, if necessary, R by its multiple and X by the intersection of $n-2$ general divisors from $|R|$, one can reduce the problem to the case when X is a surface (in which case an algebraic proof of the Hodge index theorem was given by B. Segre, J. Bronowski, A. Grothendieck, and O. Zariski). Thus it seems that Theorem 2.1 should be true if X is nonsingular in codimension two. Also, under some mild assumptions, the theorem holds in the case when R is a Weil (and not necessarily a Cartier) divisor. On the other hand, Ch. Peskine told me that the Hodge index theorem for surfaces can be deduced from Castelnuovo's bound for (possibly singular and nonreduced) curves, and so the relationship between this theorem and bounds for classes considered in this section might turn out to be deeper than it looks. In fact, I recently found that both (a very general form of) Hodge index theorem and Castelnuovo inequality are consequences of a general theory yielding an upper bound for the dimension of the ambient space of primitive families of intersecting linear subspaces.

(vi) After proving Theorem 2.1 I learned that various versions of it have been repeatedly rediscovered by various authors under different guises; cf. [Laz, 1.6.A] for a thorough discussion and, more specifically, [Laz, Corollary 1.6.3] for an equivalent statement).

The following example illustrates Theorem 2.1 in a very simple case.

Example 2.3. Let $X = Q \subset \mathbb{P}^3$ be a nonsingular quadric, let ℓ_1 and ℓ_2 be its generators and let $R \sim a_1\ell_1 + a_2\ell_2$. Then $(R^2) = 2a_1a_2 \le \frac{(a_1+a_2)^2}{2} = \frac{r_1^2}{d}$ with equality holding if and only if $a_1 = a_2 = a$, i.e. $R \sim aH$.

The next example plays a crucial role in the present paper.

Example 2.4. Let $X \subset \mathbb{P}^N$ be a nondegenerate n-dimensional variety, let $L \subset \mathbb{P}^N$, $\dim L = N - n - 1$ be a general linear subspace, and let $p_L \colon X \to \mathbb{P}^n$ be the projection with center L. Then p_L is a finite map of degree $d = \deg X$, and we denote by $R_L \subset X$ its (apparent) ramification locus

$$R_L = \overline{\{x \in \operatorname{Sm} X \mid T_{X,x} \cap L \neq \varnothing\}},$$

where $\operatorname{Sm} X = X \setminus \operatorname{Sing} X$ is the locus of nonsingular points of X and $T_{X,x}$ denotes the tangent space to X at x. It is clear that R_L is a (Weil) divisor in X and that, as L varies, the corresponding divisors R_L are rationally equivalent to each other and there are no points in $\operatorname{Sm} X$ common to all R_L. If, furthermore, X is smooth, then it is easy to determine the ambient linear system $|R_L|$. To wit, $K_{\mathbb{P}^n} = \mathcal{O}(-(n+1))$, and if ω is a rational rank n differential form on $\mathbb{P}^n$, then $(p_L^*(\omega)) \sim -(n+1)H + R_L$, where H is the divisor of a hyperplane section of X, and so

$$|R_L| = |K_X + (n+1)H|$$

(here $K_{\mathbb{P}^n}$ (resp. K_X) denotes the canonical class of $\mathbb{P}^n$ (resp. X)). In this case the linear system $|R_L|$ is ample (cf. [Za, Chapter I, Corollary 2.14]) and even very ample on X (cf. [Ein]). Thus it is natural to call the number (R_L^n) *ramification degree* (i.e. the degree with respect to the embedding defined by ramification divisor). By definition, if $X_i = X \cap \mathbb{P}^{N-n+i}$ is the i-dimensional section of X by a general linear subspace $\mathbb{P}^{N-n+i} \supset L$, then $R_L(X_i) = R_L(X) \cap \mathbb{P}^{N-n+i}$ and $(R_L(X_i)^i) = (R_L^i H^{n-i}) = r_i$ (cf. Remark 2.2 (i)). Theorem 1.1 applies to $R = R_L$ and yields $r_i = (R_L^i H^{n-i}) \le \frac{r_1^i}{d^{i-1}}$. Let $C = X_1 \subset \mathbb{P}^{N-n+1}$ be a general curve section of X, so that $r_1 = r_1(X) = r_1(C)$. Suppose that C is nonsingular, and let π be the genus of C. Then $\deg C = \deg X = d$ and, by the above, $r_1(X) = r_1(C) = \deg K_X + 2H = 2\pi - 2 + 2d$ (this is just the Riemann-Hurwitz formula). Furthermore, $\pi \le \mathcal{C}(d,a)$, where $a = N - n = \operatorname{codim} X = \operatorname{codim} C$ and

$$\mathcal{C}(d,a) = \frac{(d-\varepsilon)(d-a+\varepsilon-2)}{2a}, \qquad \varepsilon \equiv d \pmod{a}, \quad 1 \le \varepsilon \le a$$

is the Castelnuovo bound (cf. e.g. [G-H, p. 252]. The function

$$\phi(t) = \frac{(d-t)(d-a+t-2)}{2a}$$

attains maximal value for $t = \frac{a}{2} + 1$ and $\phi\left(\frac{a}{2}+1\right) = \frac{(d-\frac{a}{2}-1)^2}{2a}$. Thus

$$\mathcal{C}(d,a) \le \frac{(d-\frac{a}{2}-1)^2}{2a},$$

$$r_1 = 2\pi - 2 + 2d \le \frac{(d-\frac{a}{2}-1)^2}{a} + 2d - 2 = \frac{(d+\frac{a-2}{2})^2}{a},$$

$$r_i \le \frac{r_1^i}{d^{i-1}} \le d\left(\frac{(d+\frac{a-2}{2})^2}{ad}\right)^i.$$

Corollary 2.5. *Let $X \subset \mathbb{P}^N$ be a nondegenerate nonsingular n-dimensional variety of codimension $a = N - n$ and degree d, let $R = R_L \sim K_X + (n+1)H$ (cf. Example 2.4), and let $1 \le i \le n$. Then $r_i = (R^i H^{n-i}) \le \frac{r_1^i}{d^{i-1}} < d\left(\frac{d}{a} + \frac{5}{4}\right)^i$. Furthermore, if $d \ge \frac{(a-2)^2}{8}$ (which is always true provided that $a \le 12$), then $r_i < d\left(\frac{d}{a}+1\right)^i$.*

Corollary 2.6. *Let $X \subset \mathbb{P}^N$ be a nondegenerate nonsingular n-dimensional variety of degree d, codimension $a = N - n$ and sectional genus π, let K_X be the canonical class of X, let $R = R_L \sim K_X + (n+1)H$, and let $r_1 = \deg K_X = 2\pi + 2d - 2$.*

(i) *One has:*

$$\deg(K_X^i) = (K_X^i H^{n-i}) \le d\left(\frac{r_1}{d} - n - 1\right)^i = \frac{(r_1 - d(n+1))^i}{d^{i-1}}$$
$$= \frac{(2\pi - (n-1)d - 2)^i}{d^{i-1}} < d\left(\frac{d}{a} + \frac{1}{4} - n\right)^i, \quad 1 \le i \le n.$$

If, moreover, $d \ge \frac{(a-2)^2}{8}$ (which is always true provided that $a \le 12$), then $\deg(K_X^i) < d\left(\frac{d}{a} - n\right)^i$, $1 \le i \le n$.

(ii) *Suppose that $\pi \le \frac{(n-1)d}{2} + 1$. Then $K_X^i H^{n-i} \le 0$ for all $1 \le i \le n$.*

(iii) *If $\pi \le \frac{(n-1)d}{2} + 1$ (resp. $\pi < \frac{(n-1)d}{2} + 1$), then no positive multiple of the canonical class K_X can be a positive divisor, so that either $K_X = 0$ or $P_m = P_m(X) = l(mK_X) = 0$ for all $m > 0$ (resp. $P_m = 0$ for all $m > 0$). If $\pi \ge \frac{(n-1)d}{2} + 1$ (resp. $\pi > \frac{(n-1)d}{2} + 1$), then no negative multiple of the canonical class can be a positive divisor, so that either $K_X = 0$ or $l(-mK_X) = 0$ for $m > 0$ (resp. $l(-mK_X) = 0$ for $m > 0$); in particular, $-K_X$ cannot be ample.*

(iv) *Suppose that $d \le an + 2$. Then X satisfies the assumption of* (ii), *so that, in particular, no multiple of the canonical class K_X can be a positive divisor. Furthermore, if $d \le an + 1$, then $P_m = P_m(X) = 0$ for all $m > 0$;*

(v) *Suppose that $d < an + \frac{(an)^{\frac{n-1}{n}}}{n+1} + 2$. Then the canonical class K_X cannot be ample on X.*

Remarks **2.7**.

(i) Sometimes the bounds in Example 2.4 and Corollary 2.5 can be improved. For example, if $a = N - n$ is odd, then $\operatorname{argmax} \phi = \left[\frac{a}{2}\right] + 1$ and one gets slightly better upper bounds for r_1 and r_n. Thus, if $a = 1$, then $r_1 \le \frac{(d-1)(d-2)}{2} + 2(d-1) = d(d-1)$, $r_n \le \frac{d^n(d-1)^n}{d^{n-1}} = d(d-1)^n$, which is the best possible bound (it is sharp if and only if the hypersurface X is smooth; cf. Theorem 1.16 below) and $(K_X^i) \le d(d-n-2)^i$ which is again sharp. If $a = 2$, then $r_1 \le \frac{d^2}{2}$, $r_n \le \frac{d^{2n}}{2^n d^{n-1}} = \frac{d^{n+1}}{2^n}$ and $(K_X^i) \le d\left(\frac{d-2n-2}{2}\right)^i$, and if $a = 3$, then $r_1 \le \frac{d(d+1)}{3}$, $r_n \le d\left(\frac{d+1}{3}\right)^n$ and $(K_X^i) \le d\left(\frac{d-3n-2}{3}\right)^i$. We already saw that if $a \le 12$ or, more generally, $d \ge \frac{(a-2)^2}{8}$, then $r_n \le d\left(\frac{d}{a}+1\right)^n$ and $(K_X^i H^{n-i}) \le d\left(\frac{d}{a} - n\right)^i$. If, on the other hand, $d \le 2a+1$, then from the Clifford theorem it follows that $r_1 \le 2(2d - a - 2)$ (cf. [GH, p. 252]), Theorem 2.1 shows that $r_n \le 3^n d$, and from Corollary 2.6 it follows that K_X cannot be effective for $n \ge 2$.

(ii) After completing this work I learned that, under certain assumptions, Di Gennaro [DG] obtained Castelnuovo type bounds for (K_X^n) in terms of degree, dimension and codimension. In particular, he had to assume that K_X is nef. Apart from being intimately connected with other important invariants, such as classes (cf. Theorem 2.8 below), R_L, unlike the canonical class, is always ample (cf. Example 2.4).

(iii) It might be useful to extend Example 2.4 to more general finite coverings of $\mathbb{P}^n$ (not necessarily corresponding to projections); cf. [Laz, 6.3.D].

We apply the above results to bound the classes of projective varieties in terms of their codimension and degree.

Theorem 2.8. *Let $X \subset \mathbb{P}^N$ be an n-dimensional nonsingular variety, let R be the ramification divisor (cf. Example 2.4), let $0 \le i \le n$, and let μ_i be the i-th class of X. Then $\mu_i \le r_i$.*

Examples 2.9.

(i) Let X be a nonsingular curve of genus g. Then $r_1 = \deg R = \deg K + 2H = d^*$, and we get the *Riemann-Hurwitz formula*:

$$\mu = d^* = 2g + 2d - 2.$$

(ii) Let $X \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of degree d. Then, in the notations of 1.4, $K_X = (d-n-2)H$ and $R = K_X + (n+1)H = (d-1)H$. Thus, by Theorem 2.8, $\mu_i \le r_i = \frac{r_1^i}{d^{i-1}} = d(d-1)^i$. It is easy to see that the above inequality is actually an equality; cf. Theorem 2.14 below.

(iii) Let X be a (nonsingular) cubic scroll in $\mathbb{P}^4$. It is easy to see that $\mu_2 = \mu_2(X) = d^* = 3$ (in fact, X^* is the projection of a Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^{5*}$ from an exterior point), $\mu_1 = \mu_1(X) = 4$ and $\mu_0 = \mu_0(X) = 3$. On the other hand, $R \sim H + F$, where H is a hyperplane section and F is a fiber of X, and so $r_2 = (R^2) = 5$ and $r_1 = 4$. The reason why $\mu_2 < r_2$ is that the double locus of the projection of X from a general point $z \in \mathbb{P}^4$ is a conic $D_z \sim H - F$ containing two pinch points corresponding to the two tangent lines to D_z passing through the point z: $3 = 5 - 2$.

(iv) A general projection X'' of the Veronese surface $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ in $\mathbb{P}^3$ is called *Steiner* or *Roman surface*. The surface X'' can also be viewed as the projection of the surface $X' \subset \mathbb{P}^4$ from a point $z \in \mathbb{P}^4 \setminus X'$, where X' is a nonsingular projection of X in $\mathbb{P}^4$. As noted first by Castelnuovo, z is contained in a unique trisecant line of X' and any two of the three intersection points of this line with X' lie on a unique conic corresponding to a line in $\mathbb{P}^2$. Thus the double locus D'_z of the projection from z consists of three conics, so that X'' is singular along a union of three lines meeting in a triple point (the image of the trisecant line). In this example $R \sim D'_z \sim 3\ell$, where ℓ is the image of a line in $\mathbb{P}^2$ on $X \simeq X'$, and it is easy to see that $\mu_2(X) = 3$ (determinant of symmetric 3×3-matrix is a cubic form), $\mu_1(X) = 6$ and $\mu_0(X) = 4$ while $r_2(X) = 9$. The reason why $\mu_2 < r_2$ is that each of the three double lines on the Steiner surface obviously contains two pinch points (since any point in the complement of a plane conic is contained in exactly two tangent lines to the conic): $3 = 9 - 6$.

General formulas of this type for surfaces in $\mathbb{P}^3$ were first obtained by Salmon, Cayley and Zeuthen.

Corollary 2.10. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate nonsingular variety of codimension $a = N - n$, degree d and sectional genus π. Then*

$$d^* \le r_{n-\operatorname{def} X} \le \frac{r_1^{n-\operatorname{def} X}}{d^{n-\operatorname{def} X - 1}} < d\left(\frac{d}{a} + \frac{5}{4}\right)^{n-\operatorname{def} X}, \qquad r_1 = \mu_1 = 2\pi + 2d - 2.$$

If, moreover, $d \geq \dfrac{(a-2)^2}{8}$, *then* $d^* \leq d\left(\dfrac{d}{a}+1\right)^{n-\operatorname{def} X}$.

Example 2.11. Let $X^n \subset \mathbb{P}^N$ be a nonsingular rational normal scroll. Then $d = \deg X = N-n+1$ and $\operatorname{def} X = n-2$ (since each tangent hyperplane to X contains a linear generator $\mathbb{P}^{n-1} \subset X$ and the corresponding hyperplane section is reducible, so that the dimension of its singular locus, equal to the defect, is $n-2$). Another way to see this is to use the fact that X is a linear section of a Segre variety $\mathbb{P}^1 \times \mathbb{P}^a \subset \mathbb{P}^{2a+1}$ which is easily seen to be self-dual. Then it is easy to show that X^* is a projection of the Segre variety, and so $\dim X^* = a+1 = N-n+1$ and $\operatorname{def} X = N-1-(N-n+1) = n-2$. The same argument shows that $d^* = \deg X^* = a+1 = N-n+1 = d$. On the other hand, in this case $r_1 = 2d-2$ and Corollary 2.10 only gives $d^* \leq \dfrac{r_1^2}{d} = 4d-8-\dfrac{4}{d}$ which fails to be sharp for $d \neq 2$. It should be noted that for $d=3$ this example reduces to Example 1.14 (iii) and that any smooth section of X is again a rational normal scroll.

Corollary 2.12. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate nonsingular variety of degree d and codimension a. Then* $\mu_i \leq \dfrac{\mu_1^i}{d^{i-1}} < d\left(\dfrac{d}{a}+\dfrac{5}{4}\right)^i$. *If, moreover,* $d \geq \dfrac{(a-2)^2}{8}$, *then* $\mu_i < d\left(\dfrac{d}{a}+1\right)^i$.

Remarks **2.13.**

(i) The first inequality in Corollary 2.10 actually yields a bound for the codegree of X in terms of the degree d and the sectional genus $\pi(X)$ (defined as $\pi(X) = g(X_{n-1})$). In the case when the sectional genus of X is much less than the maximum given by Castelnuovo's theorem this bound is much better than the general one given by the second inequality.

(ii) The bounds for codegree given in Corollaries 2.10 and 2.12 are not optimal. The main reason for the failure of these bounds to be sharp is that the inequality $\mu_i \leq r_i$ proved in Theorem 2.8 is *always* strict provided that $i>1$ and $a>1$ (cf. Examples 2.9; one can give a lower bound for the difference $r_i - \mu_i$ in terms of other projective invariants). Also the bound $r_i \leq \frac{r_1^i}{d^{i-1}}$ obtained in Theorem 2.1 is not always sharp. Varieties of positive defect have an additional reason for the failure of this bound to be sharp (cf. Example 2.11). Finally, the last bounds in Corollaries 2.10 and 2.12 depend on the bound $\mathcal{C}(d,a) < d\left(\frac{d}{a}+\frac{5}{4}\right)$ (cf. Remark 2.7 (i)). The real import of Corollary 2.10 is that $\mu_i \leq \dfrac{d^{i+1}}{a^i} + \cdots$, where dots stand for an (easily computable) polynomial of degree i in d. Below we will give examples of series of varieties whose classes have the form $\mu_i = \dfrac{d^{i+1}}{a^i} + \cdots$), and so in this sense our bound for classes is good. Using Castelnuovo theory, one can prove better bounds for classes and classify the varieties on the boundary (cf. section 4).

One case when the bounds in Corollaries 2.10 and 2.12 are *is* sharp is the case of hypersurfaces (cf. Example 2.9 (ii)).

Theorem 2.14. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate (not necessarily nonsingular) variety of degree d. Then $\mu_i \leq d(d-1)^i$, $0 \leq i \leq n$ with equality (for some i) holding if and only if either $i=0$ or X is a hypersurface with $\dim(\operatorname{Sing} X) < n-i$ (i.e. X_i is a nonsingular hypersurface).*

Similarly, $d^* \leq d(d-1)^{n-\operatorname{def} X}$ with equality holding if and only if X is a cone over a nonsingular hypersurface $Y \subset \mathbb{P}^{N-\operatorname{def} X}$ with vertex $\mathbb{P}^{\operatorname{def} X-1}$ in which case $n = N-1$, $\dim Y = n - \operatorname{def} X$, $\operatorname{Sing} X = \mathbb{P}^{\operatorname{def} X-1}$ and $X^* = Y^* \subset \mathbb{P}^{N-\operatorname{def} X\,*} \subset \mathbb{P}^{N\,*}$ is the linear subspace orthogonal to the vertex of the cone X.

In particular, one always has $d^* \leq d(d-1)^n$ with equality holding if and only if X is a nonsingular hypersurface.

Remark **2.15.** For X nonsingular and $a = \operatorname{codim} X > 1$ Theorem 2.8 combined with Remarks 2.7 (i) and 2.13 (ii) yield a much better bound for classes than Theorem 2.14, viz. $\mu_i < r_i \leq \dfrac{d^{i+1}}{2^i}$.

Theorem 2.14 has a fargoing generalization to varieties of given codimension $a \geq 1$.

Definition 2.16. Let $X \subset \mathbb{P}^N$ be a projective variety, and let $\mathcal{I}_X$ be the sheaf of ideals defining X. We say that X is defined by equations of degree (not exceeding) d if the sheaf $\mathcal{I}(d)$ is generated by its global sections.

It is clear that if $d' \geq d$ and X is defined by equations of degree d, then X is also defined by equations of degree d'.

For example, a complete intersection of a hypersurfaces of degrees $d_1 \leq d_2 \leq \cdots \leq d_a = d$ is defined by equations of degree d (of course, if $d_1 < d$ and we wish to represent X (scheme theoretically) as an intersection of hypersurfaces of degree d, then we need more than a hypersurfaces).

Theorem 2.17. *Let $X^n \subset \mathbb{P}^N$, $N = n + a$ be a (not necessarily nondegenerate or nonsingular) variety defined by equations of degree d, and let i, $0 \le i \le n$ be an integer. Then $\mu_i \le \binom{a+i-1}{i} d^a (d-1)^i$ with equality holding if and only if X is a complete intersection of a hypersurfaces of degree d with $\dim(\operatorname{Sing} X) < n - i$ (so that X_i is a nonsingular complete intersection). In particular, $d^* \le \binom{N-1}{n} d^{N-n}(d-1)^n \sim \binom{N-1}{n} d^N$ with equality holding if and only if X is a nonsingular complete intersection of $N - n$ hypersurfaces of degree d.*

3. Bounds for Betti numbers

The results of section 1 largely reduce bounding of Betti numbers of nonsingular complex affine and projective varieties to bounding their classes. In section 2 we bounded the classes in terms of codimension, degree and sectional genus. In this section we combine the results of the first two sections to obtain bounds for Betti numbers.

Theorem 3.1. *Let $V^n \subset \mathbb{A}^N$ be a smooth complex affine variety, let X be the closure of V in $\mathbb{P}^N \supset \mathbb{A}^N$, let $d = \deg X$, $a = \operatorname{codim} X = N - n$, and let $b(V)$ denote the total Betti number of V. Then $b(V) < \dfrac{d^{n+1}}{a^n} + \cdots$, where dots stand for a polynomial of degree n in d. More precisely, $b(V) < a\left(\dfrac{d}{a} + \dfrac{5}{4}\right)^{n+1}$ and also $b(V) < a\left(\dfrac{d}{a} + 1\right)^{n+1} + c$, where c is a constant depending only on codimension a (and dimension n).*

Theorem 3.2. *Let X be a nonsingular complex projective variety of dimension n and degree d. Then:*

(i) *Put $t = \dfrac{r_1}{d}$, where $r_1 = \mu_1 = 2\pi + 2d - 2$ and π is the sectional genus of X. Then $b_i(X) < d\left(\dfrac{t^{i+2}}{t^2 - 1} - 1\right) < d\dfrac{t^{i+2}}{t^2 - 1}$;*

(ii) *$b_i(X) = b_{2n-i}(X) < \dfrac{d^{i+1}}{a^i} + \cdots$, where $a = \operatorname{codim} X$ and dots stand for a polynomial of degree i in d, $i \le n$. More precisely, $b_i < a\left(\dfrac{d}{a} + \dfrac{5}{4}\right)^{i+1}$ and also $b_i < a\left(\dfrac{d}{a} + 1\right)^{i+1} + c$, where c is a constant depending only on codimension a (and dimension n).*

Theorem 3.3. *Let X be a nonsingular complex projective variety of dimension n, and let $d = \deg X$. Then:*

(i) *Put $t = \dfrac{r_1}{d}$, where $r_1 = \mu_1 = 2\pi + 2d - 2$ and π is the sectional genus of X. Then $b(X) < \dfrac{dt^{n+2}}{(t-1)^2} = \dfrac{r_1 t^{n+1}}{(t-1)^2}$;*

(ii) *$b(X) < \dfrac{d^{n+1}}{a^n} + \cdots$, where $a = \operatorname{codim} X$ and dots stand for a polynomial of degree n in d. More precisely, $b(X) < \dfrac{a^2}{d}\left(\dfrac{d}{a} + \dfrac{5}{4}\right)^{n+2}$ and also $b(X) < \left(1 + \dfrac{a}{d}\right)\left(\dfrac{d}{a} + 1\right)^{n+1} + c = \dfrac{a^2}{d}\left(\dfrac{d}{a} + 1\right)^{n+2} + c$, where c is a constant depending only on codimension a (and dimension n).*

Corollary 3.4. *The total Betti number of a nonsingular complex algebraic variety, either affine or projective, does not exceed $\dfrac{d^{n+1}}{a^n} + \cdots$, where n is the dimension, a is the codimension, d is the degree of the projective closure, and dots stand for a polynomial of degree n in d.*

Remark **3.5.**

(i) The bounds in Theorems 3.1 and 3.3 are asymptotically equivalent. This means that if a variety has big degree and total Betti number, then the contribution to the total Betti number of all homologies except the middle one is negligible.

(ii) Actually the bounds in Theorems 3.1 and 3.3, (ii) and the bound for b_n in Theorem 3.2, (ii) are asymptotically sharp. However, for $i \ne n$ the bounds in Theorem 3.2, (ii) fail to be asymptotically sharp. For example, $b_0(X) = 1$ while the bound in Theorem 3.2, (ii) has the form $d + \cdots$. If $1 < i < n$, the situation is similar. If the bound in Theorem 3.2, (ii) were asymptotically sharp, then, by Corollary 2.12, the sectional genus of X would be close to maximum, so that by the Halphen-Harris theorem (cf. Theorem 4.1) X would be a divisor on a variety of minimal degree . Thus from Theorem 1.5 and Corollary 1.6 it would follow that $b_i(X) \le 1$, a contradiction.

Example 3.6. Let $X^n \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of degree d. The dual variety X_i^* is the image of X under the Gauss map defined by the partial derivatives of the equation of X_i, hence $\mu_i = d(d-1)^i$, $i = 0, \dots, n$. Since X can be viewed as a hyperplane section of the d-th Veronese embedding of $\mathbb{P}^{n+1}$, Theorem 1.5 shows that $b_i(X) = b_i(\mathbb{P}^{n+1}) = \begin{cases} 1, & 0 \le i \le 2n,\ i \equiv 0 \pmod 2,\ i \ne n, \\ 0, & 0 \le i \le 2n,\ i \equiv 1 \pmod 2,\ i \ne n \end{cases}$. Thus $e(X) = \begin{cases} b(X), & n \equiv 0 \pmod 2, \\ b(X) - 2b_n(X), & n \equiv 1 \pmod 2 \end{cases}$, and by Theorem 1.10, $e(X) = d\sum_{i=0}^{n}(-1)^i(n-i+1)(d-1)^i$. This allows to compute the numbers $b_n(X)$ and $b(X)$:

$$b_n(X) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2} = d\left(\sum_{i=0}^{n}(-1)^i\binom{n+2}{i}d^{n-i}\right) + \frac{(2n+1)(-1)^{n+1}+1}{2}.$$

and

$$b(X) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + n + 1 + (-1)^n = d\left(\sum_{i=0}^{n}(-1)^i\binom{n+2}{i}d^{n-i}\right) + (1 + (-1)^{n+1})(n+1).$$

Theorem 3.7. *Let $X^n \subset \mathbb{P}^N$ be a nonsingular n-dimensional complex projective variety of degree d. Then $b_n(X) \le b_n(\boldsymbol{X})$, where $\boldsymbol{X} \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree d. Furthermore, $b_n(X) = b_n(\boldsymbol{X})$ if and only if X is itself a hypersurface.*

Theorem 3.8. *Let $X^n \subset \mathbb{P}^N$ be a nonsingular complex projective variety of degree d. Then $b(X) \le b(\boldsymbol{X})$, where $\boldsymbol{X}$ is a smooth complex hypersurface of degree d. Furthermore, if $b(X) = b(\boldsymbol{X})$, then X is itself a hypersurface (we recall that all smooth complex hypersurfaces of dimension n and degree d have the same Betti number $b(\boldsymbol{X})$).*

Theorem 3.3 also yields a bound not involving codimension. In particular, for a nonsingular complex projective variety X of degree d one has $b(X) < \dfrac{(d+1)^{n+2}}{d}$. However, in this case Theorem 3.8 allows to get a better bound that is easy to memorize.

Theorem 3.9. *Let $X^n \subset \mathbb{P}^N$ be a nonsingular complex projective variety of degree d. Then $b(X) < d^{n+1}$.*

Building on Corollary 1.9 and Theorem 2.17, one can prove the following result extending Theorem 3.9 to varieties of arbitrary codimension defined by equations of degree d.

Theorem 3.10. *Let $X^n \subset \mathbb{P}^N$ be a (not necessarily nondegenerate) variety defined by equations of degree d. Then $b(X) < \binom{N-1}{n}d^N$.*

Example 3.11. Let $X^n \subset \mathbb{P}^N$ be a nonsingular complete intersection of hypersurfaces of degree d. Then $b_i(X) = \begin{cases} 0, & i \ne n,\ i \equiv 1 \pmod 2, \\ 1, & i \ne n,\ i \equiv 0 \pmod 2 \end{cases}$, $b_n(X) = \dots$ is computed in [Hi] and [De].

From Theorem 3.10 and Stirling's formula we derive the following bound valid for arbitrary smooth subvarieties of $\mathbb{P}^N$.

Corollary 3.12. *Let $X \subset \mathbb{P}^N$ be a nonsingular variety defined by equations of degree $\le d$. Then* $b(X) < \binom{N-1}{[\frac{N-1}{2}]}d^N \sim \sqrt{\dfrac{2}{\pi N}} \cdot (2d)^N$.

Remark **3.13.** In view of Example 3.11, the bounds in Theorem 3.10 and Corollary 3.12 and a similar bound for affine varieties that easily follow from Theorems 1.3 and Theorem 2.17 are asymptotically sharp. In particular, they are better than the bounds obtained in [Mil2] and [Th] (cf. e.g. [Mil2, Theorem 2 and Corollaries 1–3]); furthermore, our bounds take into account the codimension of X. On the other hand, although Theorem 2.17 holds for arbitrary varieties, to apply the theory developed in section 1 we, unlike Milnor and Thom, need to assume that X is nonsingular.

4. Further results and open problems

As pointed out in Remark 2.13 (ii), the bounds for classes and Betti numbers obtained in Corollary 2.12 and Theorems 3.1–3.4 are not optimal. In this section we improve these bounds and describe the varieties on the boundary. We will use the following result (cf. [H-E, Theorem 3.15] or [Ci, Theorem 3.4]).

Theorem 4.1 (Halphen-Harris). *Let $C \subset \mathbb{P}^r$, $r = a+1$ be a nondegenerate curve of genus g and degree $d > 2r$ such that*

$$g > \frac{(d-\varepsilon)(d+\varepsilon-r)}{2r} + \left[\frac{\varepsilon}{r}\right], \qquad \varepsilon \equiv d \pmod r, \quad 1 \le \varepsilon \le r.$$

Then C is contained in a surface S of minimal degree in $\mathbb{P}^r$: $C \subset S \subset \mathbb{P}^r$, $\deg S = r-1$.

Using the Halphen-Harris theorem, one can prove the following

Theorem 4.2. *Let $X \subset \mathbb{P}^N$ be a nonsingular nondegenerate variety of dimension n, codimension $a = N-n$ and degree $d > 2a+2$, let b_i, $i = 0, \dots, 2n$ and $b = \sum_{i=0}^{2n} b_i$ be its Betti numbers, and let μ_i, $i = 0, \dots, n$ denote its classes (cf. Definition 1.11).*

(i) *Suppose that $\mu_i > d\left(\frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d}\right)^i$ (since $d > 2a+2$, this condition is satisfied provided that $\mu_i > d\left(\frac{d}{a+1} + \frac{9}{8}\right)^i$) for some $i > 0$. Then $X \subset V$, where $V \subset \mathbb{P}^N$, $\dim V = n+1$, $\deg V = a$ is a variety of minimal degree.*

(ii) *Suppose that $b > d\phi\left(\frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d}\right)$, where, as in the proof of Corollary 2.14, $\phi(t) = \frac{t^{n+2}}{(t-1)^2}$ (this condition is satisfied provided that $b > \frac{(a+1)^2}{d}\left(\frac{d}{a+1} + \frac{9}{8}\right)^{n+2}$). Then $X \subset V$, where $V \subset \mathbb{P}^N$, $\dim V = n+1$, $\deg V = a$ is a variety of minimal degree.*

Theorem 4.2 largely reduces (at least for big d) the general problem of bounding classes and Betti numbers to the special case of subvarieties of codimension one in varieties of minimal degree. Since the structure of varieties of minimal degree is well known (with a couple of exceptions, they are rational normal scrolls), this last problem is not hard to solve, and thus, at least when the degree is large enough, one can get both sharp bounds for classes and Betti numbers and nice bounds generalizing Theorems 2.14 and 3.9 to varieties of arbitrary codimension. We do not go into details here.

Remark **4.3.**

(i) Taking an irreducible smooth subvariety of the form $X \sim \left[\frac{d}{a}\right] H + \left(d - a\left[\frac{d}{a}\right]\right) F$, where $V^{n+1} \subset \mathbb{P}^N$ is a rational normal scroll (of degree a), and H and F are, respectively, its hyperplane section and fiber $\mathbb{P}^n \subset V$, one sees that the bounds for classes $\mu_i(X)$ and Betti numbers $b_n(X)$ and $b(X)$ given in sections 2 and 3 are "asymptotically" best possible. However this is not the case for the bounds for the other Betti numbers $b_i(X)$. As an example, consider the case $i = 1$. In fact, if the sectional genus of X is high enough, then from the Halphen-Harris theorem it follows that X is a codimension one subvariety in a variety of minimal degree, from which and the Lefschetz theorem it is easy to infer that X is regular provided that $n > 1$.

(ii) Let $b_-(X) = \sum_{i \ne n} b_i(X) = b(X) - b_n(X)$ be the sum of all Betti numbers of X except the middle one. Then from the results of section 3 it follows that $b_-(X) \le b(X') + b_{n-1}(X) \le b(X') + b_{n-1}(X') < 2b(X')$ is bounded by a polynomial of the form $dP(\frac{d}{a})$, where P is an explicitly given polynomial of degree $n-1$ with leading coefficient 2 (if d is large with respect to a, then one can simply take $P(t) = 2t^{n-1}$). On the other hand, in view of (i), it is impossible to give a bound of this form for the middle Betti number. In other words, if the degree and the total Betti number of a variety are large, then the contribution of all homologies except the middle one to the total Betti number is much less than that of the middle homology. More generally, if $b_{-l} = \sum_{|i-n|>l} b_i$, then $b_{-l}(X) = b_{-l}(X_l)$, where $X_l = X \cap L$, $L \subset \mathbb{P}^N$, $\dim L = N-l$ is a general linear subspace, and, arguing as above, we see that, for d large with respect to a, $b_{-l}(X) \le 2d\left(\frac{d}{a}\right)^{n-l-1}$.

Here we only briefly discuss some extensions and generalizations of the above results. First of all, while we considered only complex algebraic varieties, using the l-adic etale cohomology theory, most of the results can be extended to varieties over an arbitrary algebraically closed field. On the other hand, most of our results (except those for individual Betti numbers) also extend to real algebraic varieties; this is the topic of a joint paper in preparation with V. Kharlamov [Kh-Za].

Our method of proving the bounds for Betti numbers in section 3 is ultimately based on Castelnuovo's bound for curves (cf. Example 2.4) from which we derive, using the Hodge index theorem, bounds for the degrees of selfintersections of ramification divisor and classes. Thus it is desirable to get better understanding of these basic tools. It turns out that both Castelnuovo's bound and Hodge's theorem are consequences of a general theory yielding an upper bound for the dimension of the ambient space of primitive families of intersecting linear subspaces (cf. Remark 2.2, (v)). This topic will be dealt with elsewhere.

Classes, ramification degrees, and Betti numbers are only special examples of *numerical invariants* of algebraic varieties. Important examples of such invariants are given by *Chern numbers* $c_I(X)$, where I is a multiindex of weight n (e.g. $c_1^n = \pm(K^n)$ and $c_n = e$) and *Hodge numbers* $h^{p,q}$, $0 \le p,q \le n$, $p+q=n$. For each of these (and other) numerical invariants, one can consider the problem of finding its maximal value on the class of nonsingular nondegenerate projective varieties of given dimension n, codimension a and degree d and describing the varieties on the boundary. Also, it is desirable to study the relationship between these invariants. Using the theory of Chern classes of nef vector bundles and a more detailed study of polar classes, we show that, at least for d large enough, maximization of all the above numerical invariants leads to codimension one subvarieties of varieties of minimal degree. Furthermore, for maximal varieties $|c_I| \sim \dfrac{d^{n+1}}{a^n}$ (regardless of I) and $h^{p,n-p} \sim \alpha_p \dfrac{d^{n+1}}{a^n}$, where α_p is the volume of the p-th slice of the unit $(n+1)$-dimensional cube by integral hyperplanes orthogonal to the main diagonal. Moreover, the quotients $\dfrac{|c_I|}{b_n}$ (resp. $\dfrac{h^{p,n-p}}{b_n}$) are arbitrarily close to 1 (resp. to α_p) as soon as b_n is sufficiently large with respect to $\left(\dfrac{d}{a}\right)^n$.

We proceed with listing a few open problems.

1) It is desirable to extend our results to varieties with arbitrary singularities. Unfortunately, this is not easy, particularly in the case of Betti numbers, because no Lefschetz or Morse theory for singular varieties good for our purposes seems to be available. Castelnuovo theory for singular curves is as good as for nonsingular ones provided that the geometric genus is replaced by the arithmetic one. In higher dimensions, the geometric genus is a birational invariant, and so Harris did not have to assume nonsingularity in [Har]. However, it seems harder to deal with the Hodge numbers $h^{p,q}$ for $p,q>0$.
2) As pointed out in Remark 1.11, it would be nice to get a proof of Lefschetz's hard theorem "on the level of complexes" which would strengthen it just as Theorem 1.3 strengthens Lefschetz's weak theorem.
3) As pointed out in Remark 4.3 (ii), the bound for the Betti number b_i of a nondegenerate nonsingular complex variety $X^n \subset \mathbb{P}^N$ obtained in Theorem 3.2, (ii) fails to be asymptotically sharp if $i<n$. In fact, while, by Lefschetz's theorem $b_i(X)$ is dominated by $b_i(X_k)$ for $k \ge i$, one usually has a strict inequality $b_i(X_i) > b_i(X)$. Thus it is desirable to obtain sharp (or asymptotically sharp) inequalities for the individual Betti numbers other than the middle one and classify the varieties on the boundary. In particular, one would like to bound the irregularity of complex projective varieties of given dimension, codimension and degree. By Lefschetz's theory, this problem reduces to the case of surfaces. There are reasons to believe that the bound in question is roughly twice better than Castelnuovo's bound for curves.

   More generally, one would like to get sharp bounds for the Hodge numbers $h^{p,q}$ with $p+q=i<n$.
4) It is desirable to understand better the asymptotic behavior of projective varieties and in particular the condition b_n being large with respect to d^n playing an important role in stydying the behavior of numerical invariants (cf. above; the behavior of invariants of 'lower weight', such as b_i for $i<n$, is probably controlled by lower powers of d, such as d^i). For example, arbitrary complete intersections of large degree clearly have this property, and it might be worthwhile to study it from the point of view of linkage.

   There is also some evidence that, asymptotically, the partition of unity by the 'Betti weights' $\dfrac{b_p}{b}$, $0 \le p \le n$ of a nonsingular real projective variety with a sufficiently large total Betti number b is also close to the partition α_p i.e. to the partition of unity by the 'Hodge weights' $\dfrac{h^{p,n-p}}{b_n}$

of the corresponding nonsingular complex projective variety (the complex conjugation duality $h^{p,n-p}(X_{\mathbb{C}}) = h^{n-p,p}(X_{\mathbb{C}})$ corresponds then to the Poincaré duality $b_p(X_{\mathbb{R}}) = b_{n-p}(X_{\mathbb{R}})$). It is desirable to understand the connection between these two theories.

5) It would be nice to find bounds for Betti numbers (and, possibly, for analogs of classes and other numerical invariants) for subvarieties of abelian varieties, subvarieties of toric varieties and, maybe, subvarieties of other special varieties.

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