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<td>Tono, Keita</td>
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Kyoto University
On a new class of rational cuspidal plane curves

Keita Tono

1 Introduction

Let $C$ be a curve on $\mathbb{P}^2 = \mathbb{P}^2(C)$. A singular point of $C$ is said to be a cusp if it is a locally irreducible singular point. We say that $C$ is cuspidal if $C$ has only cusps as its singular points. Suppose that $C$ is a rational cuspidal plane curve with $n$ cusps. The curve $C$ is said to be unicuspidal (resp. bicuspidal) if $n = 1$ (resp. $n = 2$). Let $\kappa = \kappa(\mathbb{P}^2 \setminus C)$ denote the logarithmic Kodaira dimension of the complement of $C$. By [Ts], there exist no rational cuspidal plane curves with $\kappa = 0$. Let $C'$ denote the proper transform of $C$ via the minimal embedded resolution of the cusps. If $n = 1$ and $\kappa = 2$, then $(C')^2 \leq -2$ by [Y]. If $n = 2$, then $\kappa = 2$ if and only if $(C')^2 \leq -1$ by [To1]. From these facts and the following theorem, $(C')^2$ is bounded from above if $\kappa = 2$.

**Theorem 1.** Let $C$ be a rational cuspidal plane curve with $n$ cusps. If $n \geq 3$, then $(C')^2 \leq 7 - 3n$.

For a fixed $n$, we consider the class of the curves with $\kappa = 2$ having the maximal $(C')$. We begin with the case: $n = 1$. In [O], Orevkov constructed two infinite sequences $C_{4k}$, $C_{4k}'$ ($k = 1, 2, \ldots$) of rational unic cuspidal plane curves with $\kappa = 2$ in the following way. Let $N$ be the nodal cubic. Let $T$ be one of two analytic branches at the node. Let $\phi : V \to \mathbb{P}^2$ denote $7$-times of blow-ups over the points which are infinitely near to $T$ and the node. The exceptional curve $E$ of $\phi$ is a linear chain of $6$-pieces of $(-2)$-curves and one $(-1)$-curve $E'$ as an end point. The curve $E$ intersects $N$ in two points. Let $\phi' : W \to \mathbb{P}^2$ denote the contraction of the proper transform of $N$ and the $6$-pieces of $(-2)$-curves in $E$. The curve $\phi'(E')$ is the nodal cubic. Put $f = \phi' \circ \phi^{-1}$. Let $C_0$ be the tangent line at a flex of $N$ and $C_0'$ an irreducible conic meeting with $N$ only in one smooth point. He defined the curves $C_{4k}$, $C_{4k}'$ as $C_{4k} = f(C_{4k-4})$, $C_{4k}' = f(C_{4k-4}')$ ($k = 1, 2, \ldots$). For $k \geq 2$, $T$ should be chosen as the analytic branch at the node which is not tangent to $C_{4k-4}$ (resp. $C_{4k-4}'$). They have the following properties for each $k$.

(i) $(C_{4k})^2 = (C_{4k}')^2 = -2$.

(ii) $\kappa(\mathbb{P}^2 \setminus C_{4k}) = \kappa(\mathbb{P}^2 \setminus C_{4k}') = 2$.

The following theorem characterizes the Orevkov's curves by $(C')^2$.

**Theorem 2.** Let $C$ be a rational unic cuspidal plane curve with $\kappa = 2$. Then $(C')^2 = -2$ if and only if $C$ is projectively equivalent to one of the Orevkov's curves.
We next consider the class of the curves with \( n = 2 \). For a cusp \( P \) of \( C \), we denote the multiplicity sequence of the cusp by \( m_P(C) \), or simply \( m_P \). We use the abbreviation \( m_k \) for a subsequence of \( m_P \) consisting of \( k \) consecutive \( m \)'s. For example, \( (2k) \) means an \( A_{2k} \) singularity. The set of the multiplicity sequences of the cusps of \( C \) will be called the numerical data of \( C \). For example, the rational quartic with three cusps has the numerical data \( \{(2), (2), (2)\} \).

**Theorem 3.** The numerical data of a rational bicuspidal plane curve \( C \) with \( (C'_1)^2 = -1 \) coincides with one of those in the following table, where \( a \) is a positive integer.

<table>
<thead>
<tr>
<th>No.</th>
<th>Numerical data</th>
<th>Degree</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>{((ab + b - 1, ab - 1, b_{a-1}, b - 1), ((ab)_2, b_a)} (b \geq 2)}</td>
<td>(2ab + b - 1)</td>
</tr>
<tr>
<td>2</td>
<td>{((ab + b, ab, b_a), ((ab + 1)_2, b_a)}</td>
<td>(2ab + b + 1)</td>
</tr>
<tr>
<td>3</td>
<td>{((ab + 1, ab - b + 1, b_{a-1}), ((ab)_2, b_a)} (b \geq 3)}</td>
<td>(2ab + 1)</td>
</tr>
<tr>
<td>4</td>
<td>{((ab + b, ab, b_a), ((ab + b - 1)_2, b_a, b - 1)} (b \geq 3)}</td>
<td>(2ab + 2b - 1)</td>
</tr>
</tbody>
</table>

Conversely, for a given numerical data in the above table, there exists a rational cuspidal plane curve having that data.

**Remark.** In [Fe], Fenske constructed sequences of rational bicuspidal plane curves. The numerical data of the curves with \( (C'_1)^2 = -1 \) among them coincide with the data 1, 2 and 3 with \( a = 1 \) in Theorem 3.

Now we pass to the case: \( n \geq 3 \). There are no known examples of curves with \( n \geq 5 \). There is only one known curve \( C \) with \( n = 4 \). The curve \( C \) is a quintic curve with \( (C'_1)^2 = -7 \). The bound given by Theorem 1 is the best possible one for \( n = 3 \) as the quartic curve \( C \) with three cusps satisfies \( (C'_1)^2 = -2 \). Moreover, we prove the following:

**Theorem 4.** Let \( C \) be a rational cuspial plane curve with three cusps. Then \( (C'_1)^2 = -2 \) if and only if \( C \) coincides with the quartic curve having three cusps.

## 2 Preliminary results

In this section, we prepare preliminaries for the proofs of our theorems. Let \( D \) be a reduced effective divisor with only simple normal crossings on a smooth surface. Let \( \Gamma \) denote the weighted dual graph of \( D \). We sometimes do not distinguish between \( \Gamma \) and \( D \). We define a blow-up over \( \Gamma \) as the weighted dual graph of the reduced total transform of \( D \) via the blow-up at a point \( P \in D \). The converse modification of the graph \( \Gamma \) is called the contraction of the vertex corresponding to the exceptional curve. The blow-up is called sprouting (resp. subdivisional) if \( P \) is a smooth point (resp. node) of \( D \). Let \( D_1, \ldots, D_r \) be the irreducible components of \( D \). We denote by \( d(\Gamma) \) the determinant of the \( r \times r \) matrix \((-D_iD_j)\). By convention, we set \( d(\Gamma) = 1 \) if \( \Gamma \) is empty.

Assume that \( \Gamma \) is connected and linear. Give \( \Gamma \) an orientation from an end point of \( \Gamma \) to the other. There are two such orientations if \( r > 1 \). The linear graph \( \Gamma \) together with one of the orientations is called a twig. The empty graph is, by definition, a twig. If necessary, renumber \( D_1, \ldots, D_r \) so that the
orientation of the twig $\Gamma$ is from $D_1$ to $D_r$ and $D_i D_{i+1} = 1$ for $i = 1, \ldots, r - 1$. We denote $\Gamma$ by $[-D_1^2, \ldots, -D_r^2]$. The twig is called rational if every $D_i$ is rational. In this note, we always assume that every twig is rational. The twig $\Gamma$ is called admissible if it is not empty and $D_i^2 \leq -2$ for each $i$.

Let $\mathbf{A} = [a_1, \ldots, a_r]$ be an admissible twig. The rational number $e(\mathbf{A}) := d([a_2, \ldots, a_r]) / d(\mathbf{A})$ is called the inductance of the twig $\mathbf{A}$. By [Fu, Corollary 3.8], the function $e$ defines a one-to-one correspondence between the set of all the admissible twigs and the set of rational numbers in the interval $(0, 1)$. For a given admissible twig $\mathbf{A}$, the admissible twig $\mathbf{A}^*$ with $e(\mathbf{A}^*) = 1 - e([a_r, \ldots, a_1])$ is called the adjoint of $\mathbf{A}$ ([Fu, 3.9]). For an integer $n$ with $n \geq 0$, we put

$$t_n = [2, \ldots, 2].$$

For non-empty twigs $\mathbf{A} = [a_1, \ldots, a_r]$, $\mathbf{B} = [b_1, \ldots, b_s]$, we write $\mathbf{A} \ast \mathbf{B} = [a_1, a_2, a_r + b_1 - 1, b_2, \ldots, b_s]$. The following lemma will be useful for computing the adjoints of admissible twigs.

**Lemma 5.** The following assertions hold true.

(i) For a positive integer $n$ and an admissible twig $\mathbf{A}$, we have $[\mathbf{A}, n + 1]^* = t_n \ast \mathbf{A}^*$.

(ii) For an admissible twig $\mathbf{A} = [a_1, \ldots, a_r]$, we have $\mathbf{A}^* = t_{a_r - 1} \ast \cdots \ast t_{a_2 - 1}$.

We will use the following lemma, which can be proved by using [Fu, Proposition 4.7].

**Lemma 6.** Let $\mathbf{A}$ be an admissible twig and $a$ a positive integer. Let $\mathbf{B}$ be a twig which is empty or admissible. Assume that the twig $[\mathbf{A}, 1, \mathbf{B}]$ is obtained from the twig $[a]$ by blow-ups $\pi$ and that $[a]$ is the image of $\mathbf{A}$ under $\pi$.

(i) There exists a positive integer $n$ such that $\mathbf{A}^* = [\mathbf{B}, n + 1, t_{a_1} \ast \cdots \ast t_{a_r}]$. Moreover, if $\mathbf{B} \neq \emptyset$, then $\mathbf{A} = [a] \ast t_n \ast \mathbf{B}^*$.

(ii) The first $n$ blow-ups of $\pi$ are sprouting and the remaining ones are subdivisional.

Conversely, for given positive integers $a$, $n$ and an admissible twig $\mathbf{B}$, the twig $[a] \ast t_n \ast \mathbf{B}^*, 1, \mathbf{B}$ shrinks to $[a]$.

## 3 Outlines of the proofs

Let $C$ be a rational cuspidal plane curve and $P_1, \ldots, P_n$ the cusps of $C$. Let $\sigma : V \to \mathbb{P}^2$ be the composite of a shortest sequence of blow-ups such that the reduced total transform $D := \sigma^{-1}(C)$ is a simple normal crossing divisor. Since $C$ is rational and cuspidal, $X := V \setminus D$ is a $\mathbb{Q}$-homology plane. Let $C'$ denote the proper transform of $C$. For each $k$, the dual graph of $\sigma^{-1}(P_k) + C'$ has the following shape.

![Diagram of the dual graph](image)
Here $D_0^{(k)}$ is the exceptional curve of the last blow-up over $P_k$ and $g_k \geq 1$.

The morphism $\sigma$ contracts $A_1^{(k)} + D_0^{(k)} + B_1^{(k)}$ to a $(-1)$-curve $E$, which is the image of $A_{g_k}^{(k)} + A_{g_k-1}^{(k)} + E + B_{g_k-1}^{(k)}$ to a $(-1)$-curve, which is the image of $A_{g_k-1}^{(k)}$, and so on. The self-intersection number of every irreducible component of $A_i^{(k)}$ and $B_i^{(k)}$ is less than $-1$ for each $i$. See [BK, MaSa] for detail. We give the graphs $A_1^{(k)}, \ldots, A_{g_k}^{(k)}$ (resp. $B_1^{(k)}, \ldots, B_{g_k}^{(k)}$) the orientation from the left-hand side to the right (resp. from the bottom to the top) in the above figure. We assign each vertex the self-intersection number of the corresponding curve as its weight. With these orientations and weights, we regard $A_i^{(k)}$ and $B_i^{(k)}$ as twigs. Let $\sigma^{(k)}$ be the composite of the blow-ups over $P_k$ of $\sigma$. There exists a decomposition $\sigma^{(k)} = \sigma_0^{(k)} \circ \sigma_1^{(k)} \circ \cdots \circ \sigma_{g_k}^{(k)}$ such that $\sigma_i^{(k)}$ contracts $[A_i^{(k)}, 1, B_i^{(k)}]$ to a $(-1)$-curve for each $i \geq 1$. Let $\rho_i^{(k)}$ denote the number of the sprouting blow-ups in $\sigma_i^{(k)}$ with respect to the $(-1)$-curve. The following lemma follows from Lemma 6.

**Lemma 7.** We have $A_i^{(k)} = \tau_i^{(k)} \ast B_i^{(k)}$, and $A_i^{(k)} = [B_i^{(k)}, \rho_i^{(k)} + 1]$.

### 3.1 Theorem 1

Let $K$ be a canonical divisor on $V$. Let $\omega_k$ (resp. $\rho_k$) be the number of the subdivisional (resp. sprouting) blow-ups of $\sigma$ over $P_k$, where the first blow-up over $P_k$ is regarded as the subdivisional one. Theorem 1 follows from the following lemma.

**Lemma 8.** Suppose $n \geq 3$. We have

$$0 \leq K(K + D) = 7 - 2n - (C')^2 - \sum_{k=1}^{n} \rho_k.$$ 

Moreover, we have $(C')^2 \leq 7 - 3n$. The equality holds if and only if $K(K + D) = 0$ and the dual graph of $\sigma^{-1}(P_k)$ is linear for each $k$.

**Proof.** We have $K(K + D) = 7 - D^2 - \sum_{k=1}^{n} (\omega_k + \rho_k)$. By [MaSa, Lemma 4], we get the desired equality. By [To3, Lemma 4.1] (cf. [BLMN, Proposition 5.8]), $0 \leq K(K + D)$. The second blow-up of $\sigma$ over $P_k$ is a sprouting one for each $k$. Hence $(C')^2 \leq 7 - 3n$. \qed

### 3.2 Theorem 2

Assume that $n = 1$, $(C')^2 = -2$ and $\bar{k} = 2$. We omit the $(k)$'s of $A_i^{(k)}, B_i^{(k)}$, etc. for the sake of simplicity. We see that one and only one of the two irreducible components of $D - D_0 - C'$ meeting with $D_0$ must be a $(-2)$-curve. Let $F_0'$ denote the $(-2)$-curve and $S_2$ the remaining one. Let $\sigma_0 : V \to V'$ be the contraction of $D_0$ and $C'$. Since $(F_0')^2 = 0$ on $V'$, there exists a $\mathbb{P}^1$-fibration $p' : V' \to \mathbb{P}^1$ such that $F_0'$ is a nonsingular fiber. Put $p = p' \circ \sigma_0 : V \to \mathbb{P}^1$. Since $k(\mathbb{P}^1 \setminus C) = 2$, there exists an irreducible component $S_1$ of $D - D_0 - F_0'$ meeting with $F_0'$ on $V$. Put $F_0 = F_0' + D_0 + C'$. The curve $S_1$ (resp. $S_2$) is a 1-section (resp. 2-section) of $p$. The divisor $D$ contains no other sections of $p$. A general fiber of $p|_X$ is isomorphic to $\mathbb{C}^{**} = \mathbb{P}^1 \setminus \{3 \text{ points}\}$. The surface
Figure 1: Dual graphs of $S_1 + S_2 + F_0 + F_1 + F_2$
Figure 2: The dual graphs of $D + E_1 + E_2$

$X = V \setminus D$ is a $\mathbb{Q}$-homology plane. Such fibrations had already been classified in [MiSu].

From [MiSu], one can deduce that $p$ has at most two singular fibers $F_1, F_2$ other than $F_0$. The fiber $F_1$ (resp. $F_2$) meets with $S_2$ in one point (resp. two points). For each $i$, let $E_i$ be the sum of all the irreducible components of $F_i$ which are not components of $D$. It follows from [MiSu] that the dual graph of $S_1 + S_2 + F_0 + F_1 + F_2$ must be one of those in Figure 1. In the figure, $\ast$ (resp. $\circ$) is a $(-1)$-curve (resp. $(-2)$-curve), $F_1 = T_{11} + E_1 + E_{12} + F_{11} + F_{12}$, $F_2 = T_{21} + E_{21} + T_{22} + E_{22} + F_{23} + T_{24}$ and $E_2 = E_{21} + E_{22}$. The divisor $T_{ij}$ may be empty for each $i, j$.

There exists a birational morphism $\varphi : V \to \Sigma_d$ from $V$ onto the Hirzebruch surface $\Sigma_d$ of degree $d$ for some $d$. The morphism $\varphi$ is the composite of the successive contractions of the $(-1)$-curves in the singular fibers of $p$. By Lemma 7, $\sigma$ gives equations on the twigs $A_i$ and $B_i$. Similar to $\sigma$, $\varphi$ gives equations on twigs for each type of the fibration $p$. One can prove that the equations for type $(\text{III}_{1a})$ and $(\text{III}_{1a}) + (\text{IV}_{2b})$ have solutions, whose weighted dual graphs coincide with those in Figure 2, where $k \geq 0$. The equations for the remaining types have no solution. From the definition of $C_{4k}$ and $C^*_{4k}$, one can show that $C$ coincides with $C_{4(k+1)}$ (resp. $C^*_{4(k+1)}$) if the fibration is of type $(\text{III}_{1a})$ (resp. $(\text{III}_{1a}) + (\text{IV}_{2b})$).

### 3.3 Theorem 3

Assume that $n = 2$ and $(C')^2 = -1$. Put $F'_0 = D_0^{(1)}$. Let $\sigma_0 : V \to V'$ be the contraction of $C'$. Since $(F'_0)^2 = 0$ on $V'$, there exists a $\mathbb{P}^1$-fibration $p' : V' \to \mathbb{P}^1$ such that $F'_0$ is a nonsingular fiber. Put $p = p' \circ \sigma_0 : V \to \mathbb{P}^1$. 

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Figure 3: The dual graph of $D + E_1 + E_2$

and $F_0 = F'_0 + C'$. Let $S_1$ and $S_2$ be the irreducible components of $A^{(1)}_{g_1} + B^{(1)}_{g_1}$ meeting with $D_0^{(1)}$. Put $S_2 = D_0^{(2)}$. The curves $S_1$, $S_2$ and $S_3$ are 1-sections of $p$. The divisor $D$ contains no other sections of $p$. A general fiber of $p|_{X}$ is isomorphic to $\mathbb{P}^1 \setminus \{3\text{ points}\}$. In the same way as in the previous case, we use the knowledge of $\mathbb{C}^{*}$-fibrations on $\mathbb{Q}$-homology planes.

From [MiSu], one can deduce that $p$ has two singular fibers $F_1$, $F_2$ other than $F_0$. For each $i$, let $E_i$ be the sum of all the irreducible components of $F_i$ which are not components of $D$. It follows from [MiSu] that the dual graph of $D + E_1 + E_2$ modulo the permutation of $S_1$ and $S_3$ must be that in Figure 3. In the figure, $F_i = T_{i1} + E_{i1} + T_{i2} + F'_i + T_{i3} + E_{i2} + T_{i4}$ and $E_i = E_{i1} + E_{i2}$ for $i = 1, 2$. The divisor $T_{ij}$ may be empty for each $i, j$. Similar to the previous case, we deal with the equations on twigs obtained from $\varphi$ and those given by Lemma 7. The weighted dual graphs of the solutions of the equations modulo the permutation of $P_1$ and $P_2$ coincide with those in Figure 4. In the figure, the graphs (1), ..., (4) correspond to curves having the numerical data 1, ..., 4 in Theorem 3, respectively.

For the proof of the converse assertion, let $\Gamma$ be one of the weighted dual graphs in Figure 4. It follows from [Fu, Proposition 4.7] that the sub-graphs $F_0$, $F_1$ and $F_2$ of $\Gamma$ can be contracted to three disjoint 0-curves. After the contraction, $S_1$, $S_2$ and $S_3$ become disjoint 0-curves and meet with each curve $F_i$ transversally. Thus $\Gamma$ can be realized by blow-ups over three sections and fibers of $\Sigma_0$. By Lemma 6, $\Gamma - E_1 - E_2 - C'$ can be contracted to two points of $\mathbb{P}^2$. Hence all the numerical data in Theorem 3 can be realized as those of rational cuspidal plane curves.

### 3.4 Theorem 4

Assume that $n = 3$. By Lemma 8, we have $(C')^2 \leq -2$. Suppose $(C')^2 = -2$. By Lemma 7 and Lemma 8, we get the following:

**Lemma 9.** The following assertions hold for each $k$.

1. $g_k = 1$ and $\rho_k^{(k)} = 1$.
2. $A_1^{(k)} = t_1 \ast B_1^{(k)}$ and $A_1^{(k)*} = [B_1^{(k)}, 2]$.

Let $\sigma_0 : V \to V'$ be the contraction of $D_0^{(2)}$ and $D_0^{(3)}$. Since $\sigma_0(C')^2 = 0$, there exists a $\mathbb{P}^1$-fibration $p' : V' \to \mathbb{P}^1$. Because $\sigma_0(D - C')\sigma_0(C') = 5$, the fibration $p'|_X$ is a $\mathbb{C}^{(*)}$-fibration. Dissimilar to the previous cases, we do not
Figure 4: The dual graph of $D + E_1 + E_2$
have the knowledge of $C^{(4+)}$-fibrations on $\mathbb{Q}$-homology planes. But one can determine the structure of the fibration by using the fact that our fibration is obtained from a rational cuspidal plane curve with three cusps. Similar to the previous cases, we deal with the equations on twigs obtained from $\varphi$ and those given by Lemma 9. By analyzing the equations, one can prove that the equations have only one solution, which corresponds to the quartic curve with three cusps.

References


E-mail address: ktono@rimath.saitama-u.ac.jp

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SAITAMA UNIVERSITY, SHIMO–OKUBO 255, URAWA SAITAMA 338–8570, JAPAN.