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On a new class of rational cuspidal plane curves

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1 Introduction

Let C be a curve on $\mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$. A singular point of C is said to be a *cusp* if it is a locally irreducible singular point. We say that C is *cuspidal* if C has only cusps as its singular points. Suppose that C is a rational cuspidal plane curve with n cusps. The curve C is said to be *unicuspidal* (resp. *bicuspidal*) if $n = 1$ (resp. $n = 2$). Let $\bar{\kappa} = \bar{\kappa}(\mathbf{P}^2 \setminus C)$ denote the logarithmic Kodaira dimension of the complement of C. By [Ts], there exist no rational cuspidal plane curves with $\bar{\kappa} = 0$. Let C' denote the proper transform of C via the minimal embedded resolution of the cusps. If $n = 1$ and $\bar{\kappa} = 2$, then $(C')^2 \leq -2$ by [Y]. If $n = 2$, then $\bar{\kappa} = 2$ if and only if $(C')^2 \leq -1$ by [To1]. From these facts and the following theorem, $(C')^2$ is bounded from above if $\bar{\kappa} = 2$.

THEOREM 1. *Let C be a rational cuspidal plane curve with n cusps. If $n \geq 3$, then $(C')^2 \leq 7 - 3n$.*

For a fixed n, we consider the class of the curves with $\bar{\kappa} = 2$ having the maximal (C'). We begin with the case: $n = 1$. In [O], Orevkov constructed two infinite sequences C_{4k}, C^*_{4k} $(k = 1, 2, \ldots)$ of rational unicuspidal plane curves with $\bar{\kappa} = 2$ in the following way. Let N be the nodal cubic. Let Γ be one of two analytic branches at the node. Let $\phi : W \to \mathbf{P}^2$ denote 7–times of blow–ups over the points which are infinitely near to Γ and the node. The exceptional curve E of ϕ is a linear chain of 6–pieces of (-2)–curves and one (-1)–curve E' as an end point. The curve E intersects N in two points. Let $\phi' : W \to \mathbf{P}^2$ denote the contraction of the proper transform of N and the 6–pieces of (-2)–curves in E. The curve $\phi'(E')$ is the nodal cubic. Put $f = \phi' \circ \phi^{-1}$. Let C_0 be the tangent line at a flex of N and C^*_0 an irreducible conic meeting with N only in one smooth point. He defined the curves C_{4k}, C^*_{4k} as $C_{4k} = f(C_{4k-4})$, $C^*_{4k} = f(C^*_{4k-4})$ $(k = 1, 2, \ldots)$. For $k \geq 2$, Γ should be chosen as the analytic branch at the node which is not tangent to C_{4k-4} (resp. C^*_{4k-4}). They have the following properties for each k.

(i) $(C'_{4k})^2 = ({C^*_{4k}}')^2 = -2$.

(ii) $\bar{\kappa}(\mathbf{P}^2 \setminus C_{4k}) = \bar{\kappa}(\mathbf{P}^2 \setminus C^*_{4k}) = 2$.

The following theorem characterizes the Orevkov's curves by $(C')^2$.

THEOREM 2. *Let C be a rational unicuspidal plane curve with $\bar{\kappa} = 2$. Then $(C')^2 = -2$ if and only if C is projectively equivalent to one of the Orevkov's curves.*

We next consider the class of the curves with $n = 2$. For a cusp P of C, we denote the *multiplicity sequence* of the cusp by $\bar{m}_P(C)$, or simply $\bar{m}_P$. We use the abbreviation m_k for a subsequence of $\bar{m}_P$ consisting of k consecutive m's. For example, (2_k) means an A_{2k} singularity. The set of the multiplicity sequences of the cusps of C will be called the *numerical data* of C. For example, the rational quartic with three cusps has the numerical data $\{(2),(2),(2)\}$.

THEOREM 3. *The numerical data of a rational bicuspidal plane curve C with $(C')^2 = -1$ coincides with one of those in the following table, where a is a positive integer.*

No.	*Numerical data*		*Degree*
1	$\{(ab+b-1, ab-1, b_{a-1}, b-1),\ ((ab)_2, b_a)\}$	$(b \geq 2)$	$2ab+b-1$
2	$\{(ab+b, ab, b_a),\ ((ab+1)_2, b_a)\}$	$(b \geq 2)$	$2ab+b+1$
3	$\{(ab+1, ab-b+1, b_{a-1}),\ ((ab)_2, b_a)\}$	$(b \geq 3)$	$2ab+1$
4	$\{(ab+b, ab, b_a),\ ((ab+b-1)_2, b_a, b-1)\}$	$(b \geq 3)$	$2ab+2b-1$

Conversely, for a given numerical data in the above table, there exists a rational cuspidal plane curve having that data.

REMARK. In [Fe], Fenske constructed sequences of rational bicuspidal plane curves. The numerical data of the curves with $(C')^2 = -1$ among them coincide with the data 1, 2 and 3 with $a = 1$ in Theorem 3.

Now we pass to the case: $n \geq 3$. There are no known examples of curves with $n \geq 5$. There is only one known curve C with $n = 4$. The curve C is a quintic curve with $(C')^2 = -7$. The bound given by Theorem 1 is the best possible one for $n = 3$ as the quartic curve C with three cusps satisfies $(C')^2 = -2$. Moreover, we prove the following:

THEOREM 4. *Let C be a rational cuspidal plane curve with three cusps. Then $(C')^2 = -2$ if and only if C coincides with the quartic curve having three cusps.*

2 Preliminary results

In this section, we prepare preliminaries for the proofs of our theorems. Let D be a reduced effective divisor with only simple normal crossings on a smooth surface. Let Γ denote the weighted dual graph of D. We sometimes do not distinguish between Γ and D. We define a blow-up over Γ as the weighted dual graph of the reduced total transform of D via the blow-up at a point $P \in D$. The converse modification of the graph is called the contraction of the vertex corresponding to the exceptional curve. The blow-up is called *sprouting* (resp. *subdivisional*) if P is a smooth point (resp. node) of D. Let $D_1, \ldots, D_r$ be the irreducible components of D. We denote by $d(\Gamma)$ the determinant of the $r \times r$ matrix $(-D_iD_j)$. By convention, we set $d(\Gamma) = 1$ if Γ is empty.

Assume that Γ is connected and linear. Give Γ an orientation from an end point of Γ to the other. There are two such orientations if $r > 1$. The linear graph Γ together with one of the orientations is called a *twig*. The empty graph is, by definition, a twig. If necessary, renumber $D_1, \ldots, D_r$ so that the orientation of the twig Γ is from D_1 to D_r and $D_iD_{i+1} = 1$ for $i = 1, \dots, r-1$. We denote Γ by $[-D_1^2, \dots, -D_r^2]$. The twig is called *rational* if every D_i is rational. In this note, we always assume that every twig is rational. The twig Γ is called *admissible* if it is not empty and $D_i^2 \leq -2$ for each i.

Let $A = [a_1, \dots, a_r]$ be an admissible twig. The rational number $e(A) := d([a_2, \dots, a_r])/d(A)$ is called the *inductance* of the twig A. By [Fu, Corollary 3.8], the function e defines a one-to-one correspondence between the set of all the admissible twigs and the set of rational numbers in the interval $(0, 1)$. For a given admissible twig A, the admissible twig A^* with $e(A^*) = 1 - e([a_r, \dots, a_1])$ is called the *adjoint* of A ([Fu, 3.9]). For an integer n with $n \geq 0$, we put $t_n = [\overbrace{2, \dots, 2}^{n}]$. For non-empty twigs $A = [a_1, \dots, a_r]$, $B = [b_1, \dots, b_s]$, we write $A * B = [a_1, \dots, a_{r-1}, a_r + b_1 - 1, b_2, \dots, b_s]$. The following lemma will be useful for computing the adjoints of admissible twigs.

LEMMA 5. *The following assertions hold true.*

(i) *For a positive integer n and an admissible twig A, we have $[A, n+1]^* = t_n * A^*$.*

(ii) *For an admissible twig $A = [a_1, \dots, a_r]$, we have $A^* = t_{a_r - 1} * \cdots * t_{a_1 - 1}$.*

We will use the following lemma, which can be proved by using [Fu, Proposition 4.7].

LEMMA 6. *Let A be an admissible twig and a a positive integer. Let B be a twig which is empty or admissible. Assume that the twig $[A, 1, B]$ is obtained from the twig $[a]$ by blow-ups π and that $[a]$ is the image of A under π.*

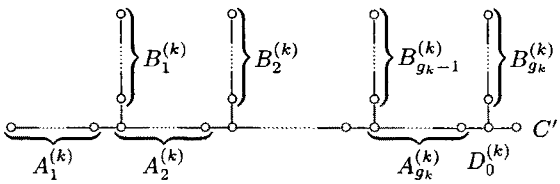
(i) *There exists a positive integer n such that $A^* = [B, n+1, t_{a-1}]$. Moreover, if $B \neq \emptyset$, then $A = [a] * t_n * B^*$.*

(ii) *The first n blow-ups of π are sprouting and the remaining ones are subdivisional.*

Conversely, for given positive integers a, n and an admissible twig B, the twig $[[a] * t_n * B^*, 1, B]$ shrinks to $[a]$.

3 Outlines of the proofs

Let C be a rational cuspidal plane curve and $P_1, \dots, P_n$ the cusps of C. Let $\sigma : V \to \mathbf{P}^2$ be the composite of a shortest sequence of blow-ups such that the reduced total transform $D := \sigma^{-1}(C)$ is a simple normal crossing divisor. Since C is rational and cuspidal, $X := V \setminus D$ is a $\mathbf{Q}$-homology plane. Let C' denote the proper transform of C. For each k, the dual graph of $\sigma^{-1}(P_k) + C'$ has the following shape.



Here $D_0^{(k)}$ is the exceptional curve of the last blow-up over P_k and $g_k \geq 1$. The morphism σ contracts $A_{g_k}^{(k)} + D_0^{(k)} + B_{g_k}^{(k)}$ to a (-1)-curve E, which is the image of $A_{g_k}^{(k)}$, $A_{g_k-1}^{(k)} + E + B_{g_k-1}^{(k)}$ to a (-1)-curve, which is the image of $A_{g_k-1}^{(k)}$, and so on. The self-intersection number of every irreducible component of $A_i^{(k)}$ and $B_i^{(k)}$ is less than -1 for each i. See [BK, MaSa] for detail. We give the graphs $A_1^{(k)}, \ldots, A_{g_k}^{(k)}$ (resp. $B_1^{(k)}, \ldots, B_{g_k}^{(k)}$) the orientation from the left-hand side to the right (resp. from the bottom to the top) in the above figure. We assign each vertex the self-intersection number of the corresponding curve as its weight. With these orientations and weights, we regard $A_i^{(k)}$ and $B_i^{(k)}$ as twigs. Let $\sigma^{(k)}$ be the composite of the blow-ups over P_k of σ. There exists a decomposition $\sigma^{(k)} = \sigma_0^{(k)} \circ \sigma_1^{(k)} \circ \cdots \circ \sigma_{g_k}^{(k)}$ such that $\sigma_i^{(k)}$ contracts $[A_i^{(k)}, 1, B_i^{(k)}]$ to a (-1)-curve for each $i \geq 1$. Let $\rho_i^{(k)}$ denote the number of the sprouting blow-ups in $\sigma_i^{(k)}$ with respect to the (-1)-curve. The following lemma follows from Lemma 6.

LEMMA 7. *We have* $A_i^{(k)} = t_{\rho_i^{(k)}} * B_i^{(k)*}$ *and* $A_i^{(k)*} = [B_i^{(k)}, \rho_i^{(k)} + 1]$.

3.1 Theorem 1

Let K be a canonical divisor on V. Let ω_k (resp. ρ_k) be the number of the subdivisional (resp. sprouting) blow-ups of σ over P_k, where the first blow-up over P_k is regarded as the subdivisional one. Theorem 1 follows from the following lemma.

LEMMA 8. *Suppose* $n \geq 3$. *We have*

$$0 \leq K(K+D) = 7 - 2n - (C')^2 - \sum_{k=1}^{n} \rho_k .$$

Moreover, we have $(C')^2 \leq 7 - 3n$. *The equality holds if and only if* $K(K+D) = 0$ *and the dual graph of* $\sigma^{-1}(P_k)$ *is linear for each* k.

PROOF. We have $K(K+D) = 7 - D^2 - \sum_{k=1}^{n}(\omega_k + \rho_k)$. By [MaSa, Lemma 4], we get the desired equality. By [To3, Lemma 4.1] (cf. [BLMN, Proposition 5.8]), $0 \leq K(K+D)$. The second blow-up of σ over P_k is a sprouting one for each k. Hence $(C')^2 \leq 7 - 3n$. □

3.2 Theorem 2

Assume that $n = 1$, $(C')^2 = -2$ and $\bar{\kappa} = 2$. We omit the (k)'s of $A_i^{(k)}$, $B_i^{(k)}$, etc. for the sake of simplicity. We see that one and only one of the two irreducible components of $D - D_0 - C'$ meeting with D_0 must be a (-2)-curve. Let F_0' denote the (-2)-curve and S_2 the remaining one. Let $\sigma_0 : V \to V'$ be the contraction of D_0 and C'. Since $(F_0')^2 = 0$ on V', there exists a $\mathbf{P}^1$-fibration $p' : V' \to \mathbf{P}^1$ such that F_0' is a nonsingular fiber. Put $p = p' \circ \sigma_0 : V \to \mathbf{P}^1$. Since $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 2$, there exists an irreducible component S_1 of $D - D_0 - F_0'$ meeting with F_0' on V. Put $F_0 = F_0' + D_0 + C'$. The curve S_1 (resp. S_2) is a 1-section (resp. 2-section) of p. The divisor D contains no other sections of p. A general fiber of $p|_X$ is isomorphic to $\mathbf{C}^{**} = \mathbf{P}^1 \setminus \{3 \text{ points}\}$. The surface

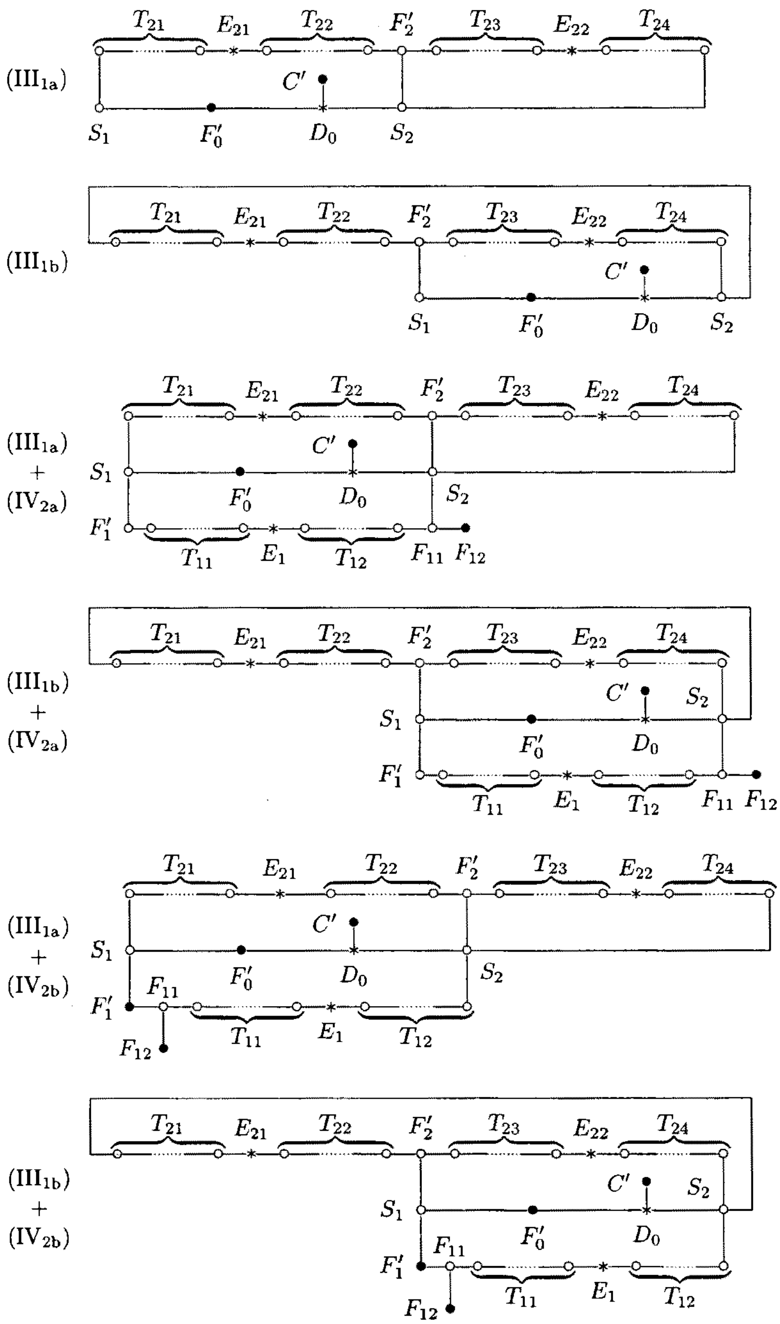
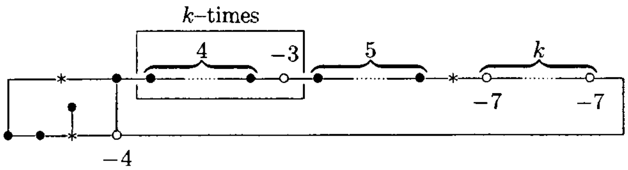
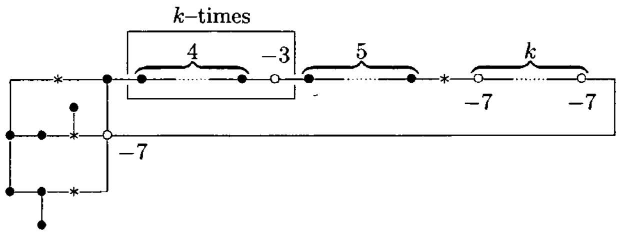


Figure 1: Dual graphs of $S_1 + S_2 + F_0 + F_1 + F_2$



(III_{1a})



$(\mathrm{III}_{1a}) + (\mathrm{IV}_{2b})$

Figure 2: The dual graphs of $D + E_1 + E_2$

$X = V \setminus D$ is a $\mathbf{Q}$–homology plane. Such fibrations had already been classified in [MiSu].

From [MiSu], one can deduce that p has at most two singular fibers F_1, F_2 other than F_0. The fiber F_1 (resp. F_2) meets with S_2 in one point (resp. two points). For each i, let E_i be the sum of all the irreducible components of F_i which are not components of D. It follows from [MiSu] that the dual graph of $S_1 + S_2 + F_0 + F_1 + F_2$ must be one of those in Figure 1. In the figure, $*$ (resp. $\bullet$) is a (-1)–curve (resp. (-2)–curve), $F_1 = T_{11}+E_1+T_{12}+F_1'+F_{11}+F_{12}$, $F_2 = T_{21} + E_{21} + T_{22} + F_2' + T_{23} + E_{22} + T_{24}$ and $E_2 = E_{21} + E_{22}$. The divisor T_{ij} may be empty for each i, j.

There exists a birational morphism $\varphi : V \to \Sigma_d$ from V onto the Hirzebruch surface Σ_d of degree d for some d. The morphism φ is the composite of the successive contractions of the (-1)–curves in the singular fibers of p. By Lemma 7, σ gives equations on the twigs A_i and B_i. Similar to σ, φ gives equations on twigs for each type of the fibration p. One can prove that the equations for type $(\mathbf{III_{1a}})$ and $(\mathbf{III_{1a}}) + (\mathbf{IV_{2b}})$ have solutions, whose weighted dual graphs coincide with those in Figure 2, where $k \geq 0$. The equations for the remaining types have no solution. From the definition of C_{4k} and C_{4k}^*, one can show that C coincides with $C_{4(k+1)}$ (resp. $C_{4(k+1)}^*$) if the fibration is of type $(\mathbf{III_{1a}})$ (resp. $(\mathbf{III_{1a}}) + (\mathbf{IV_{2b}})$).

3.3 Theorem 3

Assume that $n = 2$ and $(C')^2 = -1$. Put $F_0' = D_0^{(1)}$. Let $\sigma_0 : V \to V'$ be the contraction of C'. Since $(F_0')^2 = 0$ on V', there exists a $\mathbf{P}^1$–fibration $p' : V' \to \mathbf{P}^1$ such that F_0' is a nonsingular fiber. Put $p = p' \circ \sigma_0 : V \to \mathbf{P}^1$

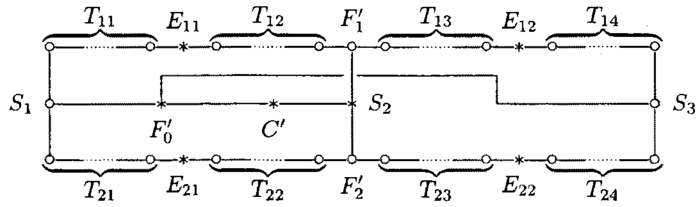


Figure 3: The dual graph of $D + E_1 + E_2$

and $F_0 = F_0' + C'$. Let S_1 and S_3 be the irreducible components of $A_{g_1}^{(1)} + B_{g_1}^{(1)}$ meeting with $D_0^{(1)}$. Put $S_2 = D_0^{(2)}$. The curves S_1, S_2 and S_3 are 1–sections of p. The divisor D contains no other sections of p. A general fiber of $p|_X$ is isomorphic to $\mathbf{C}^{**} = \mathbf{P}^1 \setminus \{3 \text{ points}\}$. In the same way as in the previous case, we use the knowledge of $\mathbf{C}^{**}$–fibrations on $\mathbf{Q}$–homology planes.

From [MiSu], one can deduce that p has two singular fibers F_1, F_2 other than F_0. For each i, let E_i be the sum of all the irreducible components of F_i which are not components of D. It follows from [MiSu] that the dual graph of $D + E_1 + E_2$ modulo the permutation of S_1 and S_3 must be that in Figure 3. In the figure, $F_i = T_{i1} + E_{i1} + T_{i2} + F_i' + T_{i3} + E_{i2} + T_{i4}$ and $E_i = E_{i1} + E_{i2}$ for $i = 1, 2$. The divisor T_{ij} may be empty for each i, j. Similar to the previous case, we deal with the equations on twigs obtained from φ and those given by Lemma 7. The weighted dual graphs of the solutions of the equations modulo the permutation of P_1 and P_2 coincide with those in Figure 4. In the figure, the graphs $(1), \dots, (4)$ correspond to curves having the numerical data $1, \dots, 4$ in Theorem 3, respectively.

For the proof of the converse assertion, let Γ be one of the weighted dual graphs in Figure 4. It follows from [Fu, Proposition 4.7] that the sub–graphs F_0, F_1 and F_2 of Γ can be contracted to three disjoint 0–curves. After the contraction, S_1, S_2 and S_3 become disjoint 0–curves and meet with each curve F_i transversally. Thus Γ can be realized by blow–ups over three sections and fibers of Σ_0. By Lemma 6, $\Gamma - E_1 - E_2 - C'$ can be contracted to two points of $\mathbf{P}^2$. Hence all the numerical data in Theorem 3 can be realized as those of rational cuspidal plane curves.

3.4 Theorem 4

Assume that $n = 3$. By Lemma 8, we have $(C')^2 \leq -2$. Suppose $(C')^2 = -2$. By Lemma 7 and Lemma 8, we get the following:

LEMMA 9. *The following assertions hold for each k.*

(i) $g_k = 1$ *and* $\rho_1^{(k)} = 1$.

(ii) $A_1^{(k)} = t_1 * B_1^{(k)*}$ *and* $A_1^{(k)*} = [B_1^{(k)}, 2]$.

Let $\sigma_0 : V \to V'$ be the contraction of $D_0^{(2)}$ and $D_0^{(3)}$. Since $\sigma_0(C')^2 = 0$, there exists a $\mathbf{P}^1$–fibration $p' : V' \to \mathbf{P}^1$. Because $\sigma_0(D - C')\sigma_0(C') = 5$, the fibration $p'|_X$ is a $\mathbf{C}^{(4*)}$–fibration. Dissimilar to the previous cases, we do not

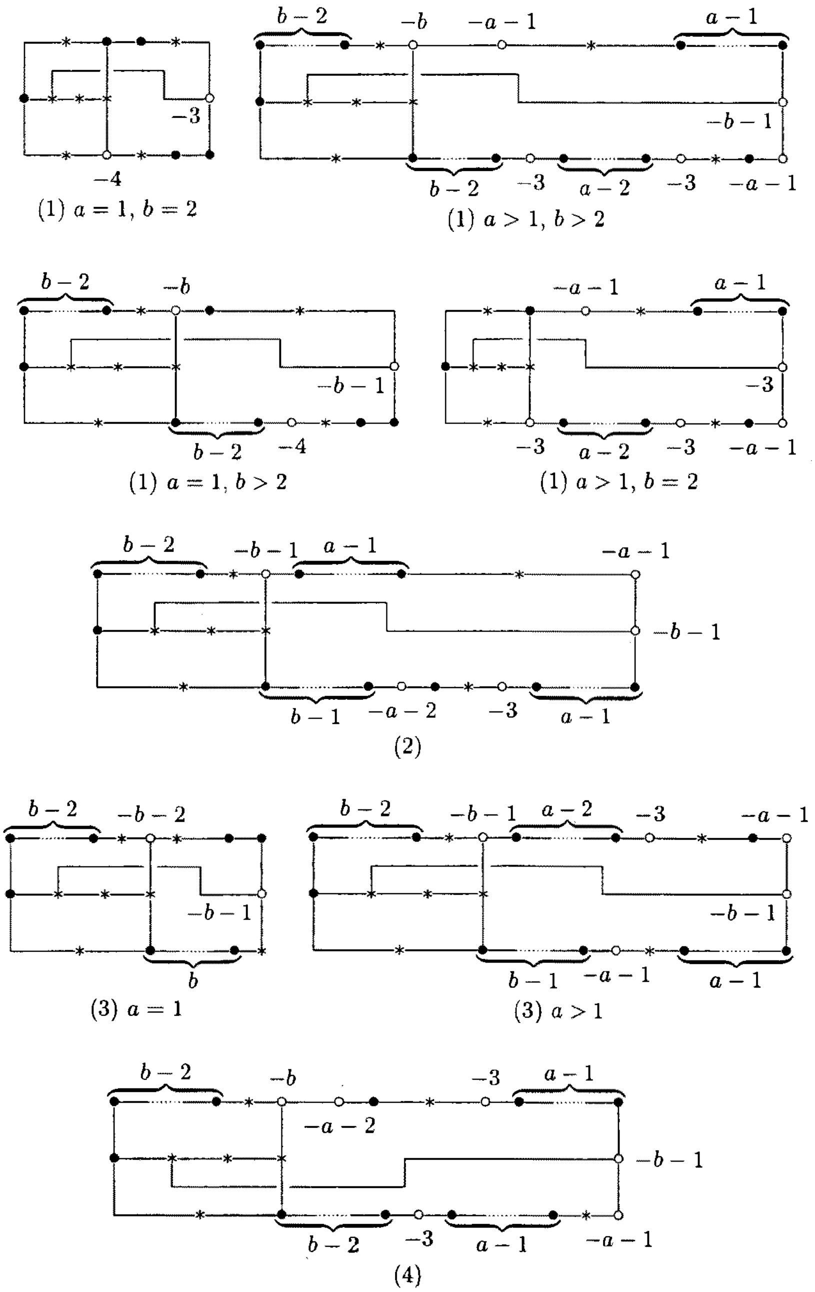


Figure 4: The dual graph of $D + E_1 + E_2$

have the knowledge of $\mathbf{C}^{(4*)}$–fibrations on $\mathbf{Q}$–homology planes. But one can determine the structure of the fibration by using the fact that our fibration is obtained from a rational cuspidal plane curve with three cusps. Similar to the previous cases, we deal with the equations on twigs obtained from φ and those given by Lemma 9. By analyzing the equations, one can prove that the equations have only one solution, which corresponds to the quartic curve with three cusps.

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