代数幾何学シンポジウム記録
2006年度 pp.19-30

PLANE MODELS OF SMOOTH PROJECTIVE CURVES

CHANGHO KEEM

ABSTRACT. We exhibit a natural relationship between the minimal degree of a plane model of a smooth projective algebraic curve X of genus g and its geometric properties; e.g. the existence of a nontrivial morphism from X onto another curve.

1. INTRODUCTION

This article is based on the talk delivered by the author at the Kinosaki Symposium 2006. There are two original research articles which are fairly closely related to what we are going to present. The first one is the article [9] jointly with G. Martens on the minimal degree of a plane model of a given algebraic curve and the other one is [3] jointly with E. Ballico on double coverings of hyperelliptic curves.

Even though some of the mathematical contents which appear in this article can also be found in the articles mentioned above, the author tries to make this article as much self-contained as possible so that the readers may obtain a reasonable overview as well as thorough details on the topics. For this reason, some part of this article may become very much similar to those in [3] and [9], which are the outcomes of joint efforts with the author's collaborators. However, the author wishes to claim that he is solely responsible for all the possible mistakes and inaccuracies in this article, if there is any.

The organization of this paper is as follows. In the next section, we start by observing a couple of examples which may provide a motivation for considering curves without plane models of small degree. The main aim of the section is to persuade the reader that the curves without plane model of small degree can be characterized as curves admitting a degree two morphism onto anther curves. In section three, we treat the variety of special linear series on double coverings of curves of low genus. Specifically, we prove that there does not exist a base-point-free and complete net of degree $g-1$ on a double covering of a curve of genus two. After mentioning the non-existence of a base-point-free and complete net of degree $g-1$, we discuss about the the primitive length of a double covering of genus two, which has been left open in one of the author's paper published long time ago. In the final section, we raise a question which is related to the theme of the second and third section.

1991 *Mathematics Subject Classification.* 14H50.

Key words and phrases. complex projective algebraic curve, double covering, gonality, Castelnuovo-Severi inequality.

Supported in part by KRF; Grant # 2005-070-C00005.

For all the notations and conventions used but not explained, we refer the reader to [2]. Otherwise stated, every curve considered in this paper is smooth irreducible and projective defined over the field of complex numbers. Throughout X is always a smooth projective curve of genus g.

2. Double coverings of low genus special curves

Long time ago (in 1884), Halphen showed that every curve X of genus g can be embedded in $\mathbb{P}^3$ as a curve of degree $g+3$ such that the hyperplane section in $\mathbb{P}^3$ is nonspecial, i.e. every curve of genus g has a nonspecial and very ample linear series of degree $g+3$; cf. [8, page 349; Proposition 6.1].

By projecting from a general point on the embedded curve $\phi(X) \subset \mathbb{P}^3$, one has a plane model X' of $X \subset \mathbb{P}^2$, which is (usually) singular.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & \phi(X) & \subset & \mathbb{P}^3 \\ & \searrow & & & \downarrow \pi \\ & & X' & \hookrightarrow & \mathbb{P}^2 \end{array} .$$

ϕ : Halphen's embedding, $\deg \phi(X) = g+3$

π : projection from a general point $p \in X$, through which there exist only a finitely many trisecant lines, whence $\pi \cdot \phi$ is birational.

$\deg X' = g+2$

Under the circumstance, we may raise the following rather naive but seemingly natural questions.

Questions 2.1. 1. What degree plane models X may have ?

2. Specifically, what is the minimal degree of a plane model X' of a given curve X ?

3. Let s_X denote the minimal degree of a plane model of X. What is the possible range of such s_X's among curves X having a fixed genus g ?

The third question can be answered easily as follows. By the genus formula for plane curves of given degree s_X, we have

$$g = g(X) \le p_a(X') = \frac{(s_X - 1)(s_X - 2)}{2}$$

where $p_a(X')$ is the arithmetic genus of the plane model X'. Therefore it follows that

$$m_0 := \frac{3 + \sqrt{8g+1}}{2} \le s_X \le g+2,$$

where the second inequality comes from the Halphen's theorem.

Having been able to obtain the interval to which s_X may belong, we further ask:

4. Does every integer in the interval $I := [m_0, g+2]$ occur as s_X for some curve X of genus g ?

5. Denoting by $\mathcal{M}_g$ the moduli space of smooth algebraic curves of genus g, we also ask: Is the natural function

$$\mathcal{M}_g \ni X \mapsto s_X \in \mathbb{N}$$

semi-continuous ?

6. What are the possible (geometric) descriptions for all those X's with a fixed s_X ?

For this series of questions, it is now fairly clear what needs to be studied. Since we are looking for morphisms $\phi : X \longrightarrow \mathbb{P}^2$ such that ϕ is generically one to one, we are indeed chasing for the so-called birationally very ample morphisms into $\mathbb{P}^2$ so that its image curve has minimal degree. This can also be realized as

$$\mathcal{D} := \{\phi^* H | H \in \mathbb{P}^{2^*}\}$$

such that

(i) for a general $p \in X$, $|\mathcal{D} - p|$ has no base point,

(ii) $\deg D \in \mathcal{D}$ is minimal.

We now begin with a couple of examples which provides a motivation for the thesis to be set up in this article.

Example 2.2. We first consider a curve which is most special in the sense of moduli. Let X be an hyperelliptic curve, i.e. $X \in \mathcal{M}_{g,2}$, where

$$\mathcal{M}_{g,2} = \{X \in \mathcal{M}_g | \ \exists\ X \xrightarrow{\pi} \mathbb{P}^1, \deg \pi = 2\}$$

We claim that $s_X = g+2$. For otherwise, take $d = s_X \leq g+1$. Then there exists g^2_d, which is birationally very ample and it follows that there also exists a base point free pencil $g^1_{d-1} = |g^2_d - p|$, $p \in X$ is a general point inducing a morphism $\psi : X \xrightarrow{g^1_{d-1}} \mathbb{P}^1$.

$$\begin{tikzcd}
 & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \\
X \arrow[ur, "\pi"] \arrow[r, "\pi\times\psi"] \arrow[dr, "\psi"'] & X' \subset \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, phantom, "\subset"] & \mathbb{P}^3 \arrow[u, "\text{projection}"'] \arrow[d, "\text{projection}"] \\
 & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1
\end{tikzcd}.$$

Since the morphism $\pi \times \psi$ is birationally very ample onto its image, X' is a curve of type $(d-1, 2)$ on a smooth quadric surface in $\mathbb{P}^3$. Hence by the adjunction formula, we have

$$g \leq p_a(X') = ((d-1)-1)(2-1) = d-2$$

which is just not compatible with the assumption

$$d = s_X \leq g+1.$$

We next claim that a smooth curve X of genus g having the minimal degree of a plane model $s_X = g+2$ must be hyperelliptic: Suppose $X \notin \mathcal{M}_{g,2}$. Then

$$\exists\, X \overset{|K_X|}{\hookrightarrow} \mathbb{P}^{g-1} \to \cdots \to X' \subset \mathbb{P}^2,$$

where $|K_X|$ the canonical embedding, followed by projections from a general point $(g-3)$ times. Therefore we have,

$$\deg X' = \deg K_X - (g-3) = g+1$$

which implies

$$s_X \leq g+1,$$

finishing the proof of the claim.

Example 2.3. On the other extreme, if X is a general curve of genus g, it was observed by Severi [12, Anhang G, §10] that

$$s_X = [\frac{2(g+4)}{3}] =: m_1.$$

In fact, this follows mainly from the Brill-Noether theorem, which was believed to be true at that time (and proved later by Griffiths and Harris in late 1970's): Denoting the variety of special linear systems of degree d and dimension r by $W^r_d(X)$, the so-called "non-existence theorem" asserts that for a general curve of genus g, $W^r_d(X) \neq \emptyset$ if and only if the Brill-Noether number $\rho(d,g,r) := g-(r+1)(g-d+r) \geq 0$. For $r=2$, one sees that m_1 is the smallest integer such that $\rho(d,g,2)$ is non-negative. Moreover, by the fact that a non-degenerate morphism correspoinding to a special g^2_d on a general curve of genus g is not composed with an involution [1] , it follows that m_1 is indeed the minimal degree of a plane model of a general curve of genus g.

Recall that, by the theorem of Halphen and the genus formula for plane curves, we have

$$m_0 := \frac{3+\sqrt{8g+1}}{2} \leq s_X \leq g+2,$$

and we asked if every integer in the interval

$$I := [m_0, g+2]$$

occur as s_X for some curve X of genus g. We now answer this question in the affirmative (at least partially) as follows.

Example 2.4. There exists a curve X of genus g with $s_X = m$ for every $m \in [m_0, m_1]$. Here we provide an outline of the proof of the existence only for the case $m \leq \frac{g+7}{2}$, which will be enough for the next Corollary 2.5, i.e., for the non-semi-continuity of the invariant s_X.

Claim: For $m \leq \frac{g+7}{2}$, $\exists\, X \in \mathcal{M}_g$ with $s_X = m$.

Let X be a smooth model of a general plane nodal curve X' of geometric genus g, $\deg X' = m$. We know that such X or X' always exists since $m \geq m_0$. Denoting by $V_{m,g}$ the Severi variety of plane curves of degree m of genus g, the plane curve

X' is indeed a general member of $V_{m,g}$. Note our numerical assumption $m \leq \frac{g+7}{2}$ is equivalent to the condition $\rho(m-3,g,1)<0$, where $\rho(d,g,r) := g-(r+1)(g-d+r)$ is the Brill-Noether number. By a result of Coppens [6], we have

$$\mathrm{gon}(X) = m-2 \qquad (*)$$

where the pencil determining the gonality is cut out by lines through a node.

While $s_X \leq m$ is trivially true, the issue here is that we may have smaller degree plane model of X. We now argue that this is not the case.

(i) Suppose $s_X \leq m-2$. By considering a pencil of lines through a general point of the minimal degree plane model, we see that $\mathrm{gon}(X) \leq m-3$, which is contradictory to the result of Coppens [6].

(ii) Suppose $s_X = m-1$. If a plane model of minimal degree s_X is singular, then $\mathrm{gon}(X) \leq m-3$ by projecting from a singular point. Hence the minimal degree plane model must be a smooth curve of degree $m-1$, whereas a smooth curve of degree $m-1$ does not have a base point free and complete g^2_m by a well known theorem of Max Noether.

As a by-product, we obtain the following corollary which answers one of our earlier questions in 2.1.

Corollary 2.5. : *The function $\mathcal{M}_g \ni X \mapsto s_X \in \mathbb{N}$ is not semi-continuous.*

Proof. If it were upper or lower semi-continuous, then the generic value m_0 achieved by a general curve of genus g should be the maximal or minimal value among all the possible value of s_X's. However, this is not the case as we have seen in the previous three examples 2.2, 2.3 and 2.4. □

Recall that Example 2.2 asserts; for $X \in \mathcal{M}_{g,2}$, $s_X = g+2$. Therefore, for a non-hyperelliptic curve

$$X \in \mathcal{M}_g \backslash \mathcal{M}_{g,2}, \text{ we have } s_X \leq g+1.$$

We now ask if the inequality $s_X \leq g+1$ for non-hyperelliptic curve X is indeed sharp. The following theorem, due to Coppens and Martens, provides an answer for the question; cf. [7, Proposition 2.2 and 2.6].

Theorem 2.6 (Coppens-Martens). *Let X be a curve of genus $g \geq 6$. Then*

$$s_X = g+1$$

if and only if

$$X \textit{ is bi-elliptic };$$

i.e. $\exists\ \pi : X \to C, \deg \pi = 2$, C an elliptic curve.

3. Double coverings of curves of low genus

Having seen that the minimal degree of a plane model of a bi-elliptic curve is $g+1$ (and that the minimal degree of a plane model of a hyperelliptic curve is $g+2$), one may further ask what is the minimal degree of a plane model of a curve which admitts a degree two finite morphism onto a curve of genus two. As for bi-elliptic curves, many things are known and we collect some of them as follows; [4, Remark (2.4.2) and Claim in p.251].

Remark 3.1. Let $\pi : X \to C$ be a double covering of a genus two curve C and assume $g \geq 11$. Then

(1)	$W^1_d(X) = W^1_6(X) + W_{d-6}(X)$	for $6 \leq d \leq g-4$
(2)	$W^2_d(X) = W^2_8(X) + W_{d-8}(X)$	for $8 \leq d \leq g-2$
(3)	$W^r_d(X) = W^r_{2r+4}(X) + W_{d-(2r+4)}(X)$	for $2r+4 \leq d \leq g-1$ and $r \geq 3$.

To improve Remark 3.1 one step further, we will prove a theorem addressing the non-existence of a base-point-free and complete net of degree $g-1$ on a double covering of a curve of genus two, which will be the essential ingredient for determining the minimal degree of a plane model of a double covering of a curve of genus two.

Theorem 3.2. *A double covering $\pi : X \to C$ of a curve of genus two with $g \geq 11$ does not carry a base-point-free and complete g^2_{g-1}. Furthermore $W^2_{g-1}(X)$ is irreducible.*

Proof. Note that for a double covering X of a curve of genus two, $\dim W^2_{g-1}(X) = g-7$ and

$$\Sigma_0 := \pi^*W^2_4(C) + W_{g-9}(X) = K - \pi^*W^2_4(C) - W_{g-9}(X) \tag{3.2.1}$$

is an irreducible component of maximal dimension; cf. [5, Corollary 2.3] and [4, Claim (ii) in p.251]. By Remark 3.1 (2), we also note that Σ_0 is the only component whose general element has a non-empty base locus. Assume the existence of a component of $W^2_{g-1}(X)$, say Σ, whose general element is base-point-free. Let $g^1_4 := \pi^*g^1_2(C)$ and we choose two sections $s_0, s_1 \in H^0(X, g^1_4)$ without common zeros. For a general $L \in \Sigma$, we consider the natural map

$$H^0(X,L) \oplus H^0(X,L) \xrightarrow{\mu} H^0(X, L \otimes g^1_4),$$

defined by $\mu(t_0,t_1) := s_0 \cdot t_0 + s_1 \cdot t_1$; $t_i \in H^0(X,L)$.

Claim: $h^0(X, L \otimes g^1_4) \geq 5$.

Proof of the Claim. If $h^0(X, L \otimes g^1_4) \leq 4$, then

$$h^0(X, L(-g^1_4)) = \dim\ker\mu = 2$$

by the base-point-free pencil trick. On the other hand, since $\deg L(-g^1_4) = g-5 \leq g-4$, $L(-g^1_4)$ is induced by π by Castelnuovo-Severi inequality, and hence

$$L(-g^1_4) = |g^1_6| + \Delta$$

for some effective divisor Δ of degree $g-11$. Then we would have $L = |g^1_4 + g^1_6| + \Delta$, contradicting L being base-point-free.

Now we consider the following two possibilities separately.

(i) $h^0(X, L\otimes g^1_4) \geq 6$: In this case, we have $h^0(X, KL^{-1}(-g^1_4)) \geq 2$ by the Riemann-Roch formula, whereas $\deg KL^{-1}(-g^1_4) = g-5 \leq g-4$. Hence by the Castelnuovo-Severi inequality and Remark 3.1, we have

$$KL^{-1}(-g^1_4) \in W^1_{g-5}(X) = \pi^*W^1_2(C) + W_{g-9}(X) \cup \pi^*W^1_3(C) + W_{g-11}(X),$$

implying

$$K - \Sigma - \{g^1_4\} \subset \pi^*W^1_2(C) + W_{g-9}(X) \cup \pi^*W^1_3(C) + W_{g-11}(X).$$

Since the locus $K - \Sigma - \{g^1_4\}$ is irreducible, we have either

$$K_X - \Sigma - \{g^1_4\} \subset \pi^*W^1_2(C) + W_{g-9}(X)$$

or

$$K - \Sigma - \{g^1_4\} \subset \pi^*(W^1_3(C)) + W_{g-11}(X).$$

If $K_X - \Sigma - g^1_4 \subset \pi^*(W^1_3(C)) + W_{g-11}(X)$, then

$$\begin{aligned} K - \Sigma &\subset \pi^*(W^1_2(C)) + \pi^*(W^1_3(C)) + W_{g-11}(X) \\ &\subset \pi^*(W^3_5(C)) + W_{g-11}(X) \subset W^3_{g-1}(X), \end{aligned}$$

which is impossible; cf. [2, Lemma 3.5, p.182]. Therefore we must have

$$K - \Sigma - \{g^1_4\} \subset \pi^*(W^1_2(C)) + W_{g-9}(X),$$

implying $K - \Sigma \subset \Sigma_0 = K - \Sigma_0$, which is again a contradiction.

(ii) $h^0(X, L\otimes g^1_4) = 5$: In this case, we have $h^0(X, L - g^1_4) = \dim\ker\mu > 0$ and hence

$$L = g^1_4 \otimes \mathcal{O}(p_1 + \cdots + p_{g-5})$$

and we may assume that $h^0(X, \mathcal{O}(p_1 + \cdots + p_{g-5})) = 1$; otherwise L would not be base-point-free by Castelnuovo-Severi inequality. We also note that among the points $p_1, \cdots, p_{g-5}$, at most one pair of points, say $\{p_{g-6}, p_{g-5}\}$, is in the same fiber of π; otherwise L is not base-point-free either.

(ii-a) $\{p_{g-6}, p_{g-5}\}$ is in the same fiber of π: In this case, $L = |\pi^*(g^1_3) + p_1 + \cdots + p_{g-7}|$. For $j = 1, \cdots, g-7$, we consider the complete linear series

$$|\pi^*(g^1_2) + p_{g-6} + p_{g-5} + p_1 + \cdots + p_j + \bar{p}_1 + \cdots + \bar{p}_j| = |\pi^*(g^{1+j}_{3+j})|,$$

which is base-point-free. Since no two p_j's (for $j = 1, \cdots, g-7$) are in the same fiber of π, $\bar{p}_j$ is not a base point of the linear series

$$|L + \bar{p}_1 + \cdots + \bar{p}_j| = |\pi^*(g^{1+j}_{3+j}) + p_{j+1} + \cdots + p_{g-7}|$$

for each $j = 1, \cdots, g-7$. Hence we have

$$\dim|L + \bar{p}_1 + \cdots + \bar{p}_j| > \dim|L + \bar{p}_1 + \cdots + \bar{p}_{j-1}|,$$

implying

$$|L + \bar{p}_1 + \cdots + \bar{p}_{g-7}| = g^{g-5}_{2g-8}$$

whose dual is a g^2_6. Since X is neither trigonal nor bi-elliptic by Castelnuovo-Severi inequality, we are done with this case.

(ii-b) No two among $\{p_1, \cdots, p_{g-5}\}$ are in the same fiber of π: We may use the same argument as (ii-a) to deduce $|L + \bar{p}_1 + \cdots + \bar{p}_{g-5}| = g^{g-3}_{2g-6}$ whose dual is a g^2_4, a contradiction. □

For the variety of linear series of dimension more than one, one may refine Remark 3.1 as follows.

Theorem 3.3. *Let $\pi : X \to C$ be a double covering of a curve of genus two and $g \geq 11$. Then*

(i) $W^2_d(X) = \pi^* W^2_4(C) + W_{d-8}(X)$ *for* $8 \leq d \leq g-1$

(ii) $W^r_d(X) = \pi^* W^r_{r+2}(C) + W_{d-(2r+4)}(X)$ *for* $2r+4 \leq d \leq g$ *and* $r \geq 3$,

which are irreducible.

Proof. (i) For $d \leq g-2$, it is clear by Remark 3.1 (2). By Theorem 3.2 and the fact that $W^2_{g-2}(X) = \pi^* W^2_4(C) + W_{g-10}(X)$, we have

$$W^2_{g-1}(X) = W^2_{g-2}(X) + W_1(X) = \pi^* W^2_4(C) + W_{g-9}(X).$$

The irreducibility of those $W^2_d(X)$'s and $W^r_d(X)$'s is also clear. (ii) is also clear. □

Corollary 3.4. *Let $\pi : X \to C$ be a double covering of a curve of genus two and $g \geq 11$. Then*

$$s_X = g.$$

Proof. By Theorem 3.2 and Remark 3.1 (2), [7, Corollary 2.5], we have $s_X = g$ for the double coverings genus two curves; in fact, the linear series $|K - \pi^* g^1_2 - p_1 - \cdots - p_{g-6}| = g^2_g$ for generically chosen $p_1, \cdots, p_{g-6} \in X$ gives a plane model of X of minimal degree. □

Theorem 3.2 has another consequence, which determines the so-called **primitive length** of double coverings of genus two curves. Recall that a complete and base-point-free linear series g^r_d on a given algebraic curve is called **primitive** if its residual series is also base-point-free. For a curve of genus $g \geq 4$, there always exists primitive linear series other than the trivial (zero and canonical) linear series. Following [4], let the primitive length $l(X)$ of X be the cardinality of the finite set of integers consisting of Clifford indices of all non-trivial primitive linear series on X. It has been shown in [4] that the primitive length is an invariant detecting double coverings; cf. [4, Theorem 3.4.1]. The results we obtained so far in this section determines the primitive length of a double covering of a curve of genus two.

Corollary 3.5. *Let X be s smooth double covering of a genus two curve C where $g \geq 11$. Then*

(i) *there is no primitive net of degree $g-1$;*

(ii) *X has primitive length* 5.

Proof. (i) By Theorem 3.3, $W^2_{g-1}(X) = \pi^*W^2_4(C) + W_{g-9}(X) = \pi^*J(C) + W_{g-9}(X)$ and hence every complete net g^2_{g-1} has non-empty base locus hence not primitive.

(ii) Remark that for a base-point-free, complete and non-primitive linear series $|D|$, there exists $p \in X$ such that $h^0(X, D+p) = h^0(X, D) + 1$ and $|D+p|$ is birationally very ample. By a result of [10], there always exist a base-point-free and complete g^1_{g-3} on X. On the other hand, since there does not exist a birationally very ample and complete g^2_{g-2} on X, any base-point-free and complete g^1_{g-3} is primitive. Likewise, any base-point-free and complete g^1_{g-2} (which we know of its existence by [10]) is primitive by Theorem 3.2. Therefore the primitive linear series g^r_d on X are complete pencils $\pi^*g^1_2, \pi^*g^1_3, g^1_{g-3}, g^1_{g-2}, g^1_{g-1}$ which have different Clifford indices. □

Corollary 3.4, Theorem 2.6 and Example 2.2 as well indicate that a curve X with big s_X is rather a special curve admitting a morphism of degree two onto a curve of small genus. Therefore, looking for curves with big s_X it seems natural to ask: Does this simple pattern observed for $s_X \geq g+1$ in Example 2.2 and Theorem 2.6 continue to hold, i.e., does $s_X = g+2-h$ imply that C is a double cover of a curve of genus (at most) h - provided that g is not too small with respect to h ? It turns out that the answer to this question is also **YES**; cf. [9].

Theorem 3.6. *Let $0 \leq t \in \mathbb{Z}$ and X be a curve of genus g with $s_X = g+2-h$. Then there is an effective polynomial expression $p(h)$ in h such that $g \geq p(h)$ implies that X is a double cover of a curve of genus at most h.*

Since the proof is somewhat involved using rather conventional (and complicated) techniques, we do not intend to provide it here. The reader is advised to look at the paper [9]. Instead we will give the proof of the following proposition which may be regarded as the converse part of Theorem 3.6. Since we want to produce a very ample linear series g^r_d on X such that r is large w.r.t d or a $g^{r'}_{d'}$ such that d' is large w.r.t r' and such that$|K_X - g^{r'}_{d'}|$ is very ample, a reasonable candidate would be $|K_X - \pi^*K_C|$.

Proposition 3.7. *Let $\pi : X \to C$ be a double covering of curves of genus g and h, respectively. If $g \geq 4h$, then $s_X \leq g+2-h$.*

Proof. For any covering $\pi : X \to C$ and any line bundle M on C, it is known ([8, II, Ex.5.1; III, Ex.4.1; IV, Ex.2.6]) that

$$\begin{aligned} H^0(X, \pi^*M) &= H^0(C, \pi_*\pi^*M) = H^0(C, \pi_*(\pi^*M \otimes_{\mathcal{O}_X} \mathcal{O}_X)) \\ &= H^0(C, M \otimes_{\mathcal{O}_C} \pi_*\mathcal{O}_X) \end{aligned}$$

and that $\det \pi_*\mathcal{O}_X \cong \mathcal{O}_C(-D)$ for a divisor D on C such that $2D$ is linearly equivalent to the branch divisor B of π (made up by the points of C over which π ramifies); in particular, the vector bundle $\pi_*\mathcal{O}_X$ on C of rank $\deg \pi$ has degree

$$-\deg D = -\frac{1}{2}\deg B = (\deg \pi)\cdot(\tilde{g}-1) - (g-1) \leq 0.$$

Moreover ([11, I, 1], if $\deg \pi = 2$, the rank two vector bundle $\pi_*\mathcal{O}_X$ splits into the line bundles $\mathcal{O}_C$ and $\det \pi_*\mathcal{O}_X$ of degree 0 resp. $2(h-1) - (g-1) = 2h - g - 1$. For a double covering $\pi : X \to C$, we thus obtain

$$H^0(X, \pi^*M) = H^0(C, M) \oplus H^0(C, M \otimes_{\mathcal{O}_C} \mathcal{O}_C(-D)),$$

and, if $\deg M < \deg D = g + 1 - 2h$, then $H^0(C, M \otimes_{\mathcal{O}_C} \mathcal{O}_C(-D)) = 0$, i.e.,

$$H^0(X, \pi^*M) = H^0(C, M). \tag{3.7.1}$$

In particular, taking $M = \omega_C$, the canonical sheaf on C, we have

$$\deg M = 2h - 2 \leq g - 2h$$

since $g \geq 4h - 2$, and so, by (3.7.1),

$$h^0(X, \pi^*K_C) = h^0(C, K_C) = h.$$

We will show that $|K_X - \pi^*K_C|$ is very ample; since this series is a complete $g^{g-3h+2}_{2g-4h+2}$ on X, we obtain, by subtracting $g - 3h \geq 0$ general points of X from it, a simple net of degree

$$(2g - 4h + 2) - (g - 3h) = g + 2 - h$$

on X proving that $s_X \leq g + 2 - h$.

In order to show that $|K_X - \pi^*K_C|$ is very ample, we need to show that

$$h^0(X, (\pi^*K_C) + P + Q) \leq h^0(\pi^*K_C)$$

for any two points P, Q on X. Let $p := \pi(P)$, $q = \pi(Q)$, $P + P' := \pi^*(p)$ and $Q + Q' := \pi^*(q)$. Then

$$(\pi^*K_C) + P + Q = \pi^*(K_C + p + q) - P' - Q'.$$

Here, by (3.7.1),

$$h^0(X, \pi^*(K_C + p + q)) = h^0(C, K_C + p + q) = 2h - h + 1 = h + 1$$

because we still have $\deg(K_C + p + q) = 2h \leq g - 2h$. Since $|K_C + p + q|$ is base point free ([8, IV, 3.2]), so is $|\pi^*(K_C + p + q)|$, and it follows that

$$\begin{aligned} h^0((\pi^*K_C) + P + Q) &= h^0(\pi^*(K_C + p + q) - P' - Q') \\ &\leq h^0(\pi^*(K_C + p + q)) - 1 = h = h^0(\pi^*K_C). \end{aligned}$$

□

Note that Corollary 3.4 is a more precise version of Proposition 3.7 for $h = 2$.

4. Epilogue

We saw in Section 1 that for a smooth curve X of genus g, the minimal degree s_X of a plane model of X lies in the interval

$$[\frac{3 + \sqrt{8g + 1}}{2}, g + 2],$$

and that every integer in the sub-interval actually occurs as s_X for some curve X of genus g. However, we still don't know if the integers in the other part of the interval really occurs as s_X for some curve X of genus g. Of course, reasonable candidates double coverings of curves of genus h, as we saw for the cases $h = 0, 1, 2$.

The following result due to Dongsoo Shin provides an answer for this question, at least partially [13].

Theorem 4.1. *Let X be a smooth irreducible curve of genus g. If X is a double cover of a smooth irreducible curve Y of genus $h \geq 2$, then*

$$g - 2h + 3 + \mathrm{Cliff}(Y) \leq s_X \leq g - 2h + 2\,\mathrm{gon}(Y),$$

where $\mathrm{Cliff}(Y)$ *and* $\mathrm{gon}(Y)$ *denote the Clifford index and the gonality of Y, respectively.*

Recall that a curve of genus g can be embedded in $\mathbb{P}^3$ as a curve of degree $g + 3$ by a theorem of Halphen. Naturally, one may want to have a more precise version of the Halphen's statement. For example, it would be nice to have a description of those curves which may be embedded in $\mathbb{P}^3$ as a curve of degree smaller than $g + 3$. A first step toward this direction was obtained by Harris [2, Exercise B, p.221], who showed the following using a theorem of Mumford [2, Theorem 5.2, p.193].

Theorem 4.2 (Harris). *Let X be a curve of genus g. If X is not hyperelliptic, trigonal or bi-elliptic, X embeds into $\mathbb{P}^3$ as a curve of degree $g + 2$.*

Let

$$\mathcal{M} := \{X \in \mathcal{M}_g \mid X \text{ is not hyperelliptic, trigonal or bi-elliptic}\},$$

be the classes of curves admitting embeddings of degree $g + 2$ into $\mathbb{P}^3$. Note that if $\mathrm{Cliff}\, X \geq 3$ then $X \in \mathcal{M}$ and hence X carries a very ample g^3_{g+2} by Theorem 4.2. However one has

$$\mathcal{M}_3 := \{X \in \mathcal{M}_g \mid \mathrm{Cliff}(X) \geq 3\} \subsetneq \mathcal{M}.$$

Therefore one can expect: $X \in \mathcal{M}_3$ **may satisfy a stronger condition, say, the existence of a very ample g^3_{g+1} on X.**

Again, the following theorem of Dongsoo Shin provides a clue for a possble answer for the quesion raised above [14]

Theorem 4.3. *Let X be a smooth projective algebraic curve of genus $g \geq 21$ which is not a curve of even gonality admitting an automorphism of order two. Suppose that $\mathrm{gon}(X) \geq 7$ and X is neither a k-sheeted cover of an elliptic curve with $k \leq 4$ nor a plane curve of degree 8. Then there exists a complete and very ample linear series of dimension 3 and degree $g + 1$ on X.*

References

[1] Su una congettura di Petri, Comment. Math. Helvetici 1981, (56) 3-A , 1-38.

[2] Arbarello, E., Cornalba, M., Griffiths P. A. and Harris J., Geometry of Algebraic Curves I, Springer Verlag, 1985.

[3] Ballico, E. and Keem, C. On double coverings of hyperelliptic curves, Journal of Pure and Applied Algebra, to appear.

[4] Coppens, M., Keem, C. and Martens, G. Primitive linear series on curves. Manuscripta Math., 1992, 77, 237-264.

[5] Cho, K., Keem, C. and Ohbuchi, A. On the variety of special linear systems of degree $g-1$ on smooth algebraic curves. International J. Math., 2002, 13 (1), 11-29.
[6] Coppens, M. The gonality of general smooth curves with a prescribed plane nodal model, Math. Ann. **289** (1991), no. 1, 89–93.
[7] Coppens, M. and Martens, G. Linear series on 4-gonal curves. Math. Nachr. 2000, 213, 35-55.
[8] Hartshorne, R. Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
[9] Keem, C. and Martens, G. Curves without plane model of small degree. Preprint.
[10] Keem, C. and Ohbuchi, A. On the Castelnuovo-Severi inequality for a double covering, Preprint.
[11] Mumford, M. Prym varieties I, Contribution to Analysis, Acad. Press. (1974), 325–355.
[12] F. Severi, *Vorlesungen über algebraische Geometrie: Geometrie auf einer Kurve, Riemannsche Flächen, Abelsche Integrale*, Johnson Reprint Corp., New York, 1968.
[13] Shin, Dongsoo, Plane models of double covers of smooth curves, preprint.
[14] Shin, Dongsoo, Very ample webs of degree $g+1$ on smooth algebraic curves of genus g, preprint.

DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA *and* NATIONAL INSTITUTE FOR MATHEMATICAL SCIENCES, DAEJEON, KOREA

E-mail address: ckeem@math.snu.ac.kr