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ABEL MAPS FOR SINGULAR CURVES

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ABSTRACT. The Abel map embeds a nonsingular projective curve in a projective algebraic group, the so-called Jacobian variety of the curve. Using the group structure we can consider higher versions of the Abel map, which carry a lot of information about the projective geometry of the curve. If the curve varies in a family, so do its Jacobian variety and the Abel map. So it is natural to ask what happens when the family degenerates to a singular curve, for instance, to a Deligne–Mumford stable curve. We will see in this talk how to construct an analogue of the Abel map that "nearly" embeds a Gorenstein curve in a generalization of the Jacobian variety. This is joint work with Caporaso (Roma Tre) and Coelho (IMPA).

1. INTRODUCTION

Let C be a smooth, projective, connected curve over an algebraically closed field k. The (isomorphism classes of) line bundles on C form an Abelian group, under the operation of tensor product. Those of degree 0 form a subgroup. This subgroup has the structure of a scheme, a fine moduli space, and is called the Jacobian variety of C, and denoted by J_C.

Given $P \in C$, we get a map,

$$A_C\colon C \longrightarrow J_C$$

sending a point Q to the line bundle $\mathcal{O}_C(P-Q)$ associated to the degree-0 divisor $P-Q$. If C is not the rational line, then A_C is an embedding.

Using the group structure of J_C we can construct higher Abel maps. Thus, for each integer $d \geq 1$ we have a map

$$A_C^d\colon S^d(C) \longrightarrow J_C,$$

where $S^d(C)$ is the symmetric product of C, the quotient of the product $C^{\times d}$ by the permutation group, or the scheme parameterizing degree-d divisors. The d-th Abel map A_C^d sends a divisor $Q_1 + \cdots + Q_d$ to the

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line bundle associated to the divisor $dP - Q_1 - \cdots - Q_d$. Of course, $A_C^1 = A_C$.

Much of the projective geometry of C is encoded in these Abel maps. In fact, the fibers of A_C^d are the complete linear systems of degree d. More precisely, given a line bundle L of degree 0, we have

$$(A_C^d)^{-1}([L]) = \mathbf{P}(H^0(C, L^*(dP))),$$

where $L^* := Hom(L, \mathcal{O}_C)$ is the dual to L. So, for instance, C is nonhyperelliptic if and only if A_C^2 is an embedding.

Notice that, by Riemann–Roch, A_C^d is a projective bundle over J_C as long as $d \geq 2g - 1$, where g is the genus of C. In fact, this is one way the Jacobian variety is constructed, as a quotient of $S^d(C)$, for $d \geq 2g - 1$, by the equivalence relation $R \subset S^d(C) \times S^d(C)$, where two divisors are considered equivalent if they give rise to isomorphic line bundles. The fact that R is itself a projective bundle over any of the factors $S^d(C)$ is essential for the quotient to exist as a scheme.

This is the basic picture for smooth curves. The question we will be concerned with in these notes is: can we generalize the Abel map for singular curves?

In general, we can consider a variety of degree-0 line bundles on a singular curve, its so-called generalized Jacobian, but it is not projective for an irreducible curve, and not even of finite type for a reducible curve. For example, if C is a nodal cubic with node P then its generalized Jacobian is isomorphic to $C - \{P\}$, which is in turn isomorphic to the multiplicative group $\mathbb{G}_m$. So, in order to consider Abel maps we need to compactify the Jacobian.

To my knowledge, Igusa [Ig], in 1956, was the first to consider such a compactification. He considered a Lefschetz pencil of curves in a smooth surface, and studied the degeneration of the Jacobian varieties along the pencil to a singular (nodal and irreducible) member. It was only ten years later that Mayer and Mumford [MM] observed that the boundary points in Igusa's compactification could be associated to degree-0, torsion-free, rank-1 sheaves, and thus the compactification could in principle be a fine moduli space.

Let now C be simply a projective curve, that is, a projective, connected, reduced, one-dimensional scheme over an algebraically closed field k. A torsion-free, rank-1 sheaf I on C is simply a coherent sheaf which has rank 1 at the generic points of C and can be embedded in the sheaf of rational functions of C. More geometrically, a torsion-free rank-1 sheaf I is simply a tensor product $I = \mathcal{I}_{\Gamma/C} \otimes L$ of an ideal sheaf of a finite subscheme $\Gamma \subset C$ with an invertible sheaf L. The degree of I is simply the difference between the degree of L and the length of Γ.

Independently of the presentation, it can be shown that the degree of I satisfies

$$\deg(I) = \chi(I) - \chi(\mathcal{O}_C),$$

where $\chi(\cdot)$ denotes the Euler characteristic.

A line bundle corresponds to an invertible sheaf, and every invertible sheaf is torsion-free of rank 1.

If C is irreducible, D'Souza [So], in 1974, constructed, using Geometric Invariant Theory, a fine moduli space $\overline{J}_C$, parameterizing torsion-free, rank-1 sheaves of degree 0 modulo (abstract) isomorphisms. The invertible sheaves form an open subset $J_C \subsetneq \overline{J}_C$.

A little later, Altman and Kleiman [AK] considered the Abel map of C: given a simple point P of C, there is a map,

$$A_C\colon C \longrightarrow \overline{J}_C,$$

sending $Q \in C$ to $\mathcal{I}_{Q/C} \otimes \mathcal{O}_C(P)$. They showed this map is an embedding if and only if $C \not\cong \mathbf{P}^1$; in particular, always, if C is indeed singular.

Altman and Kleiman considered as well higher Abel maps, with the Hilbert schemes Hilb^d_C of C replacing the symmetric products $S^d(C)$, and showed that the fibers of these Abel maps are projective spaces. In fact, they constructed $\overline{J}_C$ as a quotient of a Hilbert scheme $\mathrm{Hilb}^d(C)$, for $d >> 0$, by an equivalence relation $R \subset \mathrm{Hilb}^d(C) \times \mathrm{Hilb}^d(C)$, which is a projective bundle over any of its factors, as before.

Suppose now that C is reducible, and let $C_1, \dots, C_n$ denote its components. The generalized Jacobian, J_C, is now an infinite union of quasiprojective varieties,

$$J_C = \coprod_{|\underline{d}|=0} J^{\underline{d}}_C,$$

where $\underline{d}$ is a n-tuple of integers $(d_1, \dots, d_n)$, and $|\underline{d}| := d_1 + \cdots + d_n$, and where $J^{\underline{d}}_C$ is the open subscheme of J_C parameterizing line bundles whose restriction to C_i has degree d_i for each $i = 1, \dots, n$.

Clearly, one does not want to compactify the whole J_C. On the other hand, even though all the $J^{\underline{d}}_C$ are (noncanonically) isomorphic, one does not want to compactify just one of the components. The reason is that we want a compactification to be a degeneration of Jacobians, if the curve C is a degeneration of smooth curves. For instance, a "triangle" (the union of three independent lines in the plane) is the flat limit of elliptic curves. Since the Jacobian of an elliptic curve is isomorphic to the curve itself, one should expect that the compactification one looks for is isomorphic to the triangle, and hence has three components.

Oda and Seshadri [OSe], in 1979, used Geometric Invariant Theory to construct several compactifications of certain (disjoint) unions of finitely many $J^{\underline{d}}_C$. Which multidegrees $\underline{d}$ to consider depended on the choice of a polarization. Later, in 1982, Seshadri [Se] handled the case of a general reduced curve. (Actually, Seshadri dealt with the higher rank case as well.)

Despite some partial work by Ishida [Is], it remained a problem to deal with families of curves. After a long time, in her 1993 thesis, Caporaso [Ca] showed how to compactify the relative Jacobian over the moduli of Deligne–Mumford stable curves, by putting on the boundary line bundles on curves derived from stable curves. One year later, in his thesis, Pandharipande [P] constructed the same compactification, with the boundary points now representing torsion-free, rank-1 sheaves, as in Seshadri's [Se]. (Also, as in [Se], Pandharipande dealt with the higher rank case.) These relative compactifications have as fibers compactifications of the type dealt with by Oda and Seshadri.

The main disadvantage of these constructions for reducible curves is that they do not yield fine moduli spaces, in contrast to those for irreducible curves. One could try to mimic the construction of $\overline{J}_C$ as a quotient of Hilbert schemes modulo equivalence relations. This was done by Altman and Kleiman [AK], but it turns out that, when C is reducible, the equivalence relation is not a projective bundle as before, so we end up with an algebraic space, instead of a scheme. Or so it seemed at the time. In fact, it was shown in [Es] that the quotient is a scheme. Moreover, a part of it, depending on the choice of a polarization, and the choice of a base point of C, behaves reasonably well as a compactification. A particular case, enough for dealing with Abel maps, is described in the next section.

2. Abel maps

Let C be a projective (connected) curve (defined over an algebraically closed field k). To avoid technical difficulties, which will not be apparent below, we will assume C is Gorenstein, that is, that its dualizing sheaf is invertible.

Let I be a torsion-free, rank-1 sheaf on C of degree 0. We say that I is *stable* (resp. *semistable*) if, for every proper subcurve $Y \subset C$,

$$|\deg I_Y| < \frac{\delta_Y}{2} \quad (\text{resp. } \leq).$$

Here I_Y is the restriction of I to Y modulo torsion. (If $I = \mathcal{I}_{\Gamma/C} \otimes L$, then $I_Y = \mathcal{I}_{(\Gamma\cap Y)/Y} \otimes L|_Y$.) And δ_Y is the length of $Y \cap \overline{C - Y}$.

The usual procedure in constructing moduli spaces is to identify semistable shaves with respect to a certain equivalence relation, called sometimes the S-equivalence or the Jordan–Hölder equivalence. The motivation for doing this is to get a separated moduli space.

For instance, if a semistable sheaf I is an extension

$$0 \longrightarrow J \longrightarrow I \longrightarrow N \longrightarrow 0,$$

we can deform the extension to a trivial extension in such a way that the abstract sheaf I remains constant until it degenerates to the direct sum $J \oplus N$. Under certain circumstances, $J \oplus N$ is also a semistable sheaf. So, to get a separated moduli space one identifies I with $J \oplus N$.

It is this identification that produces coarse moduli spaces. In [Es] a different approach is taken. Instead of identifying semistable sheaves, we rule out certain semistable sheaves from being represented in our moduli space, in such a way to avoid the existence of several limits in a degeneration. For instance, we rule out a sheaf like $J \oplus N$. This is done through the following definition:

Let Q be a simple point of C. We say that a semistable sheaf I is called *Q-quasistable* if $\deg I_Y > -\delta_Y/2$ for every proper subcurve $Y \subset C$ containing Q.

The definition of quasistability has a flavor that reminds us of the choice of a fundamental domain for a lattice. I cannot make this precise though. Anyway, it can be shown:

Theorem 2.1. [Es] *There is a scheme $\overline{J}^Q_C$ that is a complete fine moduli space for Q-quasistable sheaves modulo isomorphisms.*

There is a relative version of this theorem for a family of curves endowed with a section through its smooth locus. The geometric fibers will be the $\overline{J}^Q_C$ mentioned above, whence schemes. However, the total space may only be an algebraic space.

Another word of warning: $\overline{J}^Q_C$ may not be projective. More precisely, I do not know whether $\overline{J}^Q_C$ is projective or not. Its construction is done by patching local constructions. If C is stable, there is a surjection,

$$\overline{J}^Q_C \longrightarrow \overline{P}^0_C,$$

where $\overline{P}^0_C$ denotes the fiber over the point representing C in Caporaso's relative compactification (of degree 0). The scheme $\overline{P}^0_C$ is projective, but is only a coarse moduli space, where S-equivalent semistable sheaves are identified.

One may try to define an Abel map for C just as in the irreducible case by letting

$$A\colon C \longrightarrow \overline{J}_C^Q$$

be the map sending $N \in C$ to the sheaf $\mathcal{I}_{N/C} \otimes \mathcal{O}_C(Q)$. However, for this definition to make sense, we need to verify that such sheaf is Q-quasistable. And this is not always the case. For instance, if C is the union of just two components C_1 and C_2, meeting transversally at a point, and N and P are simple points belonging to different components, then $\mathcal{I}_{N/C} \otimes \mathcal{O}_C(Q)$ is not Q-quasistable. The best we can say is:

Theorem 2.2. [Co] *The Abel map A is well-defined if C has no separating nodes. In this case, A is an embedding unless $C \cong \mathbf{P}^1$.*

(A *separating node* is a point $N \in C$ for which there is a subcurve $Y \subset C$ such that $Y \cap \overline{C - Y} = \{N\}$, the intersection being transverse.)

Proof. First of all, for a proper subcurve $Y \subset C$,

$$\deg\left((\mathcal{I}_{N/C} \otimes \mathcal{O}_C(Q))_Y\right) \geq -1, \tag{2.2.1}$$

with equality if and only if $N \in Y$ and $Q \notin Y$. On the other hand $-\delta_Y/2 \leq -1$, unless $\delta_Y = 1$. And $\delta_Y = 1$ if and only if Y intersects $\overline{C - Y}$ transversally at a unique point. Notice as well that, if $Q \in Y$, then the inequality in (2.2.1) is necessarily strict. So, $\mathcal{I}_{N/C} \otimes \mathcal{O}_C(Q)$ is Q-quasistable if C has no separating nodes.

Suppose now that C has no separating nodes. First of all, the fiber $A^{-1}(A(N))$ can be viewed as an open subscheme of $\mathbf{P}(\mathrm{Hom}(\mathcal{I}_{Q/C}, \mathcal{O}_C))$, in fact the open subscheme parameterizing injective homomorphisms. The reader may use the fine moduli property to prove this. (This is quite similar to what happens for smooth, even irreducible curves. The difference is that for these curves every nonzero homomorphism $\mathcal{I}_{Q/C} \to \mathcal{O}_C$ is necessarily injective.)

On the other hand, as A is defined, the fiber $A^{-1}(A(N))$ is closed in C, hence complete. So, $A^{-1}(A(N))$ is also closed in $\mathbf{P}(\mathrm{Hom}(\mathcal{I}_{Q/C}, \mathcal{O}_C))$, and hence

$$A^{-1}(A(N)) = \mathbf{P}(\mathrm{Hom}(\mathcal{I}_{Q/C}, \mathcal{O}_C)).$$

Now, as $A^{-1}(A(N)) \subseteq C$, either $A^{-1}(A(N))$ is scheme-theoretically a point, or $A^{-1}(A(N)) \cong \mathbf{P}^1$. We must show the latter does not happen if $C \not\cong \mathbf{P}^1$.

So, assume $C \not\cong \mathbf{P}^1$, and let us suppose, by contradiction, that there is $N \in C$ such that $A^{-1}(A(N)) \cong \mathbf{P}^1$. So, $A^{-1}(A(N))$ is a rational component of C; let us denote it by E. Let $Q_1, Q_2 \in E$ be distinct points which are simple in C. Since $A(Q_1) = A(Q_2)$, we have that $\mathcal{I}_{Q_1/C} \cong \mathcal{I}_{Q_2/C}$.

So, there is a rational function h of C which is well-defined and nonzero everywhere, except at Q_1, where it has a single zero, and at Q_2, where it has a single pole. In particular, $h|_E$ has degree 1, and whence is injective, whereas $h|_F$ is constant for any other component $F \subset C$. Of course, since $C \not\cong \mathbf{P}^1$, there are such components F. Furthermore, if $Y \subseteq \overline{C - E}$ is a connected component, then Y does not intersect E transversally, because otherwise the point of intersection would be a separating node. Since $h|_Y$ is constant, it follows that $h|_E$ takes the same value at two points, which can be infinitesimally close. This contradicts the injectivity of $h|_E$. □

What happens if C has separating nodes? The map A is not well-defined, but a "twisted version" of it is. First, we need a definition and a notation.

A proper subcurve $Z \subset C$ is called a *tail* if Z intersects $\overline{C - Z}$ transversally at a unique point. Of course, in this case, also $\overline{C - Z}$ is a tail. If Z is a tail, and we denote by N the point in $Z \cap \overline{C - Z}$, there is a unique invertible sheaf on C whose restriction to Z is $\mathcal{O}_Z(-N)$ and to $\overline{C - Z}$ is $\mathcal{O}_{\overline{C-Z}}(N)$. We denote this sheaf by $\mathcal{O}_C(Z)$.

By abuse of notation, we will let

$$\mathcal{O}_C(\sum n_Z Z) := \bigotimes \mathcal{O}_C(Z)^{\otimes n_Z},$$

where the sum runs through tails Z and the n_Z are integers.

The notation $\mathcal{O}_C(Z)$ is justified as follows: If $\mathcal{C}$ is a regular model of C over $B := \mathrm{Spec}(\mathbb{C}[[t]])$, that is, a family of curves over B whose special fiber is C and whose total space, $\mathcal{C}$, is regular, then Z can be viewed as a Cartier divisor in $\mathcal{C}$ and

$$\mathcal{O}_{\mathcal{C}}(Z)|_C \cong \mathcal{O}_C(Z).$$

Notice that, in the situation described above, $\mathcal{O}_{\mathcal{C}}(Z)$ restricts to the trivial sheaf on the generic fiber of $\mathcal{C}$ over B. In other words, $\mathcal{O}_C(Z)$ is the limit of trivial sheaves. So, it should not hurt to twist the sheaf $\mathcal{I}_{N/C} \otimes \mathcal{O}_C(Q)$ by sheaves like $\mathcal{O}_C(Z)$. This is what we can do to get a well-defined Abel map:

Theorem 2.3. [Co] *There is a well-defined map,*

$$\widetilde{A}\colon C \to \overline{J}^Q_C,$$

such that, if N is not a separating node of C, then

$$\widetilde{A}(N) = \mathcal{I}_{N/C} \otimes \mathcal{O}_C(Q) \otimes \mathcal{O}_C(-\sum Z),$$

where the sum runs over the tails $Z \subset C$ which contain N but do not contain Q.

(If N is a separating node, $\widetilde{A}(N)$ can be inferred by passage to the limit.)

The proof of the above theorem relies on a few combinatorial reasonings, and will not be given here. It can be found in [Co]. The map $\widetilde{A}$ is natural, as it extends to the universal family over the versal deformation space of (C, Q), but we will not attempt to explain this fact in more details here.

Finally, when is $\widetilde{A}$ an embedding? To answer this, we need a definition first.

A component $E \subset C$ is called a *separating line* if E intersects $\overline{C - E}$ only at separating nodes. A subcurve $Y \subset C$ is called a *tree of separating lines* if Y is connected of arithmetic genus 0, and Y intersects $\overline{C - Y}$ in separating nodes. If this is the case, it can be shown that every connected subcurve of Y is also a tree of separating lines, and, in particular, a component of Y is a separating line.

Theorem 2.4. [Co] *Let $\widetilde{A}$ be the map mentioned in Theorem 2.3. Then*

(i) $\widetilde{A}(N_1) = \widetilde{A}(N_2)$ if and only if N_1 and N_2 belong to the same maximal tree of separating lines.
(ii) $d\widetilde{A}_N$ is injective unless N belongs to a separating line.
(iii) Let $Y \subset C$ be a maximal tree of separating lines, and $N_1, \dots, N_b$ the separating nodes of $Y \cap \overline{C - Y}$. Then the spaces $d\widetilde{A}_{N_i}(T_{N_i,C})$ are one-dimensional linearly independent subspaces of $T_{\widetilde{A}(Y), \overline{J}_C^Q}$.

In particular, the theorem says that $\widetilde{A}$ is an embedding if and only if $C \not\cong \mathbf{P}^1$ and C does not contain any separating line.

At any rate, The above theorem gives a good description of the image $\widetilde{A}(C)$. It can be shown that $\widetilde{A}(C)$ is a curve of the same arithmetic genus of C. So, as smooth curves degenerate to C, their images under the Abel map in the Jacobians degenerate to $\widetilde{A}(C)$. However, even if C is stable, $\widetilde{A}(C)$ does not need to be. In fact, $\widetilde{A}(C)$ need not be even Gorenstein.

The proof of Theorem 2.4 can be found in [Co].

It would be interesting to study the composition of $\widetilde{A}$ with the map $\overline{J}_C^Q \to \overline{P}_C^0$ to Caporaso's compactification. This was, essentially, partly done in [CaEs], which heavily influenced [Co]. However, it remains to see whether the properties stated in Theorem 2.4 apply as well to this composition.

Also, it remains to see whether higher degree Abel maps can be defined. In [Co], an interesting example of a degree-2 Abel map is carried out in detail. That example benefitted from discussions with Caporaso.

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