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On a non-reflexive embedding with birational Gauss map for a projective variety

Satoru Fukasawa (Waseda University / JSPS)

1. What is the Gauss map ?

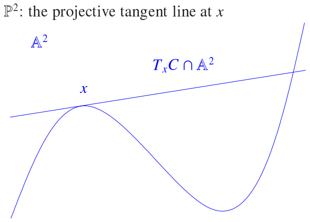
K: alg. closed field of char. $p \geq 0$

Simple Case

$C \subset \mathbb{P}^2$; plane curve

$x \in C_{sm}$; smooth point

$T_xC \subset \mathbb{P}^2$: the projective tangent line at x



T_xC=the projective closure of the tangent line (passing through x)

General Case

$X \subset \mathbb{P}^N$; projective variety

$x \in X_{sm}$

$T_xX \subset \mathbb{P}^N$; the projective tangent space at x

Gauss map $\gamma : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N) \cong \mathbb{G}(N-n-1, \mathbb{P}^{N*})$

$$x \mapsto T_xX \mapsto (T_xX)^*$$

Calculation

$x_1, \ldots, x_n$; local coordinate around $x \in X$

$\rho = (1 : x_1 : \cdots : x_n : f_{n+1} : \cdots : f_N) : X \hookrightarrow \mathbb{P}^N$

$T_xX \in \mathbb{G}(n, \mathbb{P}^N)$:

$$\begin{pmatrix} \rho \\ \partial\rho/\partial x_1 \\ \vdots \\ \partial\rho/\partial x_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & f_{n+1} & \cdots & f_N \\ 0 & 1 & 0 & \cdots & 0 & f_{n+1,x_1} & \cdots & f_{N,x_1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & f_{n+1,x_n} & \cdots & f_{N,x_n} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & f_{n+1} - \sum_{j=1}^n x_j f_{n+1,x_j} & \cdots \\ 0 & 1 & 0 & \cdots & 0 & f_{n+1,x_1} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & f_{n+1,x_n} & \cdots \end{pmatrix}.$$

Relationship between π & γ

$$\begin{tikzcd} CX \arrow[r, "\gamma'"] \arrow[d] \arrow[rr, bend left, "\pi"] \arrow[dr, phantom, "\square"] & I_{\gamma(X)} \arrow[r, "\pi'"] \arrow[d] & X^* \\ X \arrow[r, "\gamma"'] & \gamma(X) \subset \mathbb{G}(N-n-1, \mathbb{P}^{N*}) & \end{tikzcd}$$

$I_{\gamma(X)} = \{(E, H) \in \gamma(X) \times \mathbb{P}^{N*} | E \subset H\}$

$\gamma'(x, H) = (T_xX, H)$

$\dim X = 1$　　insep.deg of γ = insep.deg of π.

$\Rightarrow \pi' : I_{\gamma(X)} \to X^*$; generically smooth

(Kaji, Hefez, Voloch,...)

Kleiman-Piene's question ([7, pp.108–109])

Is $\pi' : I_{\gamma(X)} \to X^*$ separable in higher dimension ?

KPQ: affirmative $\Rightarrow$

"gen. smoothness of $\gamma \Rightarrow$ gen. smoothness of π"

($\Leftrightarrow$ reflexivity)

2. Non-reflexive embedding with birational Gauss map (NREBG)

Answer to KPQ: No !

Counterexamples of dim ≥ 3

Example (Kaji [5, 6]).

Segre variety $X = \prod_{i=1}^r \mathbb{P}^{n_i} \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^M$;

$(n := \dim X = \sum_{i=1}^r n_i.)$

- $r \geq 2 \Rightarrow \gamma : X \to \gamma(X)$ is isomorphic.
- X: non-reflexive $\Leftrightarrow \begin{cases} 2n_i < n \ for \ \forall i \\ n : odd \\ p = 2 \end{cases}$

Example (F [2]).

$p > 0$.

$Y \subset \mathbb{P}^n : X_0^{p+1} + \cdots + X_n^{p+1} = 0$

$X = Y \times \mathbb{P}^1 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^{2n+1}$

- $n \geq 2 \Rightarrow \gamma : X \to \gamma(X)$ is isomorphic.
- $n \geq 3 \Rightarrow X$: non-reflexive.

Linearity Theorem

$p = 0 \Rightarrow$ general fiber of γ: linear space.

e.g. X: curve $\Rightarrow$ general fiber of γ : 1pt.

Linearity Theorem fails in char $p > 0$!

Example.

$p > 2$.

$f : \mathbb{A}^1 \to \mathbb{P}^2; t \mapsto (1 : t : t^{2p})$

Gauss map:

$$\begin{pmatrix} 1 & x & x^{2p} \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & x^{2p} \\ 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ general fiber of γ: 2pts.　$[\gamma(x) = \gamma(-x)]$

Theorem (F [1])

$p > 0$.

$\forall Y \subset \mathbb{P}^k$ proj. variety, $\exists X \subset \mathbb{P}^N$ proj. variety s.t.

Y = general fiber of γ.

Why such phenomena occur ?

$d\gamma \equiv 0$

i.e. × The generic smoothness (Sard's Theorem) of γ

3. Reflexivity & Kleiman-Piene's question

$X \subset \mathbb{P}^N$ proj. variety of dim n

$CX := \{(x, H) \in X_{sm} \times \mathbb{P}^{N*} | T_xX \subset H\}^{-} \subset \mathbb{P}^N \times \mathbb{P}^{N*}$　conormal variety

$\pi : CX \to \mathbb{P}^{N*}$　conormal map

$X^* := \pi(CX) \subset \mathbb{P}^{N*}$　dual variety

$CX^* \subset \mathbb{P}^{N**} \times \mathbb{P}^{N*} = \mathbb{P}^N \times \mathbb{P}^{N*}$; conormal variety $/X^*$

Definition.

X is called reflexive if $CX = CX^*$.

Monge-Segre-Wallace criterion

X is reflexive $\Leftrightarrow \pi : CX \to X^*$ is generically smooth.

Proposition.

X: reflexive $\Rightarrow$ "Usual" Geometry;

(1) (Projective Duality) $X^{**} = X$.

(2) (Linearity Theorem)

general fiber of γ is linear (scheme-theoretically).

Especially, γ is generically smooth.

$\dim X = 2$

Theorem (F-Kaji [3]). Assume that $\dim X = 2$.

- γ: gen. smooth $\Rightarrow X$: reflexive
- $\exists X$ s.t. $\pi' : I_{\gamma(X)} \to X^*$: inseparable.

Next Step:

What kind of variety has non-reflexive embedding with gen. smooth Gauss map ?

Main Theorem (F-Kaji [4])

$X \subset \mathbb{P}^N$: proj. variety of dim ≥ 3, not linear.

$d\gamma \equiv 0$

$\Rightarrow \exists\tau : X \hookrightarrow \mathbb{P}^M$ s.t.

$\tau(X)$: non-reflexive & $\gamma(\tau)$: birational

$[d\gamma \equiv 0$: The differential of γ is identically zero$]$

Example (of $d\gamma \equiv 0$)

・(The graph of a Frobenius map of) the projective space $\mathbb{P}^n$

・Fermat hypersurface of degree d with $d \equiv 1$ mod p

Open Question ([6, Question 2]).

γ: gen. smooth $\Rightarrow$ gen. fibers of γ: linear ?

[1] S. Fukasawa, Varieties with non-linear Gauss fibers, Math. Ann. **334** (2006), 235–239.

[2] S. Fukasawa, On Kleiman-Piene's question for Gauss maps, Compositio Math. **142** (2006), 1305–1307.

[3] S. Fukasawa and H. Kaji, The separability of the Gauss map and the reflexivity for a projective surface, Math. Z. **256** (2007), 699–703.

[4] S. Fukasawa and H. Kaji, Existence of a non-reflexive embedding with birational Gauss map for a projective variety, to appear in Math. Nachr.

[5] H. Kaji, On the duals of Segre varieties, Geom. Dedicata **99** (2003), 221–229.

[6] 梶元, ガウス写像の非分離性に関する Kleiman-Piene の問題について, 代数幾何学シンポジウム–高次元多様体、正標数上の話題を中心として–(2003 年 1 月, 九州大学) 報告集, pp. 134–141.

[7] S. Kleiman and R. Piene, On the inseparability of the Gauss map, In: Contemp. Math. 123, Amer. Math. Soc., Providence, 1991, pp.107–129.