代数幾何学シンポジウム記録 2007年度　pp.121–121

F-thresholds on toric rings

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Abstract

This poster is written about *F-thresholds on toric rings.*
We consider some questions about them (§4)
using the formula for them on toric rings,
which is the main theorem in this poster. (§3)
In general, *F-thresholds are the invariants of pairs* $(R, \mathfrak{a})$ *of rings R of char.* $p > 0$ *and ideals* $\mathfrak{a}$.
On regular rings, they are *jumping numbers of test ideals.* (§2)
In that sense, *F-thresholds are*
analogies to jump. coeff. of multiplier ideals. (§1)

Connection between char. 0 and char. $p > 0$

— Notation in this section —

- $R_{\mathbb{Q}}$: a $\mathbb{Q}$-Goren. normal local f. g. $\mathbb{Q}$-algebra, $\mathfrak{a}_{\mathbb{Q}} \subseteq R_{\mathbb{Q}}$: an ideal.
- Assume $\exists R_{\mathbb{Z}} \subseteq R_{\mathbb{Q}}$, $\mathfrak{a}_{\mathbb{Z}} := \mathfrak{a}_{\mathbb{Q}} \cap R_{\mathbb{Z}}$ s. t. $R_{\mathbb{Z}} \otimes \mathbb{Q} \cong R_{\mathbb{Q}}$, $\mathfrak{a}_{\mathbb{Z}} R_{\mathbb{Q}} = \mathfrak{a}_{\mathbb{Q}}$. (i. e., $R_{\mathbb{Q}}$: the fibre over generic point 0.)
- $p \in \mathbb{Z}$: a prime number, $R_p := R_{\mathbb{Z}} \otimes \mathbb{F}_p$, $\mathfrak{a}_p := \mathfrak{a}_{\mathbb{Z}} R_p$. (i. e., R_p: a fibre over closed point (p).)

— Note. and Assum. about char. 0 —

$f : X' \to X := \operatorname{Spec} R_{\mathbb{Q}}$: log resolution of $\mathfrak{a}_{\mathbb{Q}}$, that is,

- f: a proper birational, X': smooth,
- $\exists D$: an effective div. on X' s. t. $\mathfrak{a}_{\mathbb{Q}}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-D)$, $\operatorname{Exc}(f) \cup \operatorname{Supp}(D)$: SNC.

finition 1 (multiplier ideals). $t \in \mathbb{R}_{>0}$. Define the multiplier ideal $\mathfrak{a}_{\mathbb{Q}}^t$) by
$$\mathcal{J}(\mathfrak{a}_{\mathbb{Q}}^t) := f_*\mathcal{O}_{X'}(\lceil K_{X'} - f^*K_X - tD \rceil),$$
ere $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, for $D = \sum d_i D_i$.

finition 2 (jumping coefficients).
$$(0 <) \xi^1(\mathfrak{a}_{\mathbb{Q}}) < \xi^2(\mathfrak{a}_{\mathbb{Q}}) < \cdots < \xi^i := \xi^i(\mathfrak{a}_{\mathbb{Q}}) < \cdots$$
the jumping coefficients of $\mathfrak{a}_{\mathbb{Q}}$, if
$$\mathcal{J}(\mathfrak{a}_{\mathbb{Q}}^{\xi^i}) = \mathcal{J}(\mathfrak{a}_{\mathbb{Q}}^t),\ \forall t \in [\xi^i, \xi^{i+1}) \subseteq \mathbb{R}_{>0}.$$

— Note. and Assum. about char. $p > 0$ —

- $F : R_p \to R_p$: the absolute Frobenius map,
- ${}^e R_p$: the ring R_p viewed as an R_p-module via F^e,
- $E := E_{R_p}$: injective hull of the residue field of R_p,
- $F_E^e : E \xrightarrow{F^e \otimes id} {}^e R_p \otimes E$.

finition 3 (test ideal). $t \in \mathbb{R}_{>0}$. Define the test ideal of $\mathfrak{a}_p^t$ by
$$\tau(\mathfrak{a}_p^t) := \operatorname{Ann}_{R_p} 0_E^{*\mathfrak{a}_p^t},$$
ere $z \in 0_E^{*\mathfrak{a}_p^t} \subseteq E \Leftrightarrow \exists c \neq 0 \in R_p$ s. t. $cF_E^e(z)\mathfrak{a}_p^{\lceil tq \rceil} = 0$, $\forall e \gg 0$.

Theorem 4 ([HY03]). For fixed $t \in \mathbb{R}_{>0}$, $\exists U \subseteq \operatorname{Spec}\mathbb{Z}$: open dense s. t.,
$$\mathcal{J}(\mathfrak{a}_{\mathbb{Q}}^t)_p \overset{\text{hom}}{=} \tau(\mathfrak{a}_p^t) \subseteq R_p, \forall p \in U.$$

We can define F-jumping coefficients as a char. p analog of jump. coeff.

Definition 5 (F-jumping coefficients).
$$(0 <) c^1(\mathfrak{a}_p) < c^2(\mathfrak{a}_p) < \cdots < c^i := c^i(\mathfrak{a}_p) < \cdots$$
are the F-jumping coefficients of $\mathfrak{a}_p$, if
$$\tau(\mathfrak{a}_p^{c^i}) = \tau(\mathfrak{a}_p^t),\ \forall t \in [c^i, c^{i+1}) \subseteq \mathbb{R}_{>0}.$$

Corollary 6 ([MTW05]).
$$\lim_{p \to \infty} c^i(\mathfrak{a}_p) = \xi^i(\mathfrak{a}_{\mathbb{Q}}),\ \forall i.$$

Question 1. $\xi^i(\mathfrak{a}_{\mathbb{Q}}) \in \mathbb{Q}$ is clear, since $K_{X'}$, f^*K_X, D: $\mathbb{Q}$-div.
Then $c^i(\mathfrak{a}_p) \in \mathbb{Q}$? (We consider this *question* at the end of this poster.)

Remark. In char. p, an another definition of F-jump. coeff. is given as the F-threshold in some rings. We give its def. and observations in the next section.

2 Connection between F-thresholds and F-jumping coefficients

— Note. and Assum. —

- R: Noether comm. reduced ring of char. $p > 0$. $\tau(R) = R$.
- R: F-pure ($\overset{\text{def}}{\Leftrightarrow}$ F splits as R-module map.)
- R: F-finite ($\overset{\text{def}}{\Leftrightarrow}$ ${}^1R$ is a f. g. R-module.)
- $\mathfrak{a}$, $J \subseteq R$: ideals s. t. $0 \neq \mathfrak{a} \subseteq \sqrt{J} \subsetneq R$.

Definition 7. Define the F-threshold of $\mathfrak{a}$ w. r. t. J by
$$c^J(\mathfrak{a}) := \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e},$$
where $\nu_{\mathfrak{a}}^J(p^e) := \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq F^e(J)\}$

Example. $R = k[[X, Y]]$, $\mathfrak{a} = (X^2 + Y^3)$, $J = (X, Y)$. Then,
$$c^J(\mathfrak{a}) = \begin{cases} \frac{5}{6} & p \equiv 1 \pmod 3, \\ \frac{5}{6} - \frac{1}{6p} & p \equiv 2 \pmod 3. \end{cases}$$

Theorem 8 ([MTW05], [BMS06]). R: a regular ring. Then,
$$c^{\tau(\mathfrak{a}^t)}(\mathfrak{a}) \leq t,\ \forall t \in \mathbb{R}_{>0},$$
$$\tau(\mathfrak{a}^{c^J(\mathfrak{a})}) \subseteq J,\ \forall J \subseteq R.$$
In particular,
$$\forall i\ c^i = c^{\tau(\mathfrak{a}^{c^i})}(\mathfrak{a}).$$

Question 2. If R: non-regular, then $\exists$ inequality bet. c^i and $c^{\tau(\mathfrak{a}^{c^i})}(\mathfrak{a})$?

Question 3. In particular, in which condition $c^i = c^{\tau(\mathfrak{a}^{c^i})}(\mathfrak{a})$?

We consider the questions on toric rings in the next section.

Remark. In [HMTW07], F-thresholds $c^J(\mathfrak{a})$ is generalized to F-thresholds $c_M^N(\mathfrak{a})$ ($N \subseteq M$: R-modules.)
$$\{c^J(\mathfrak{a}) \mid J : \text{an ideal}\} \hookrightarrow \{c_M^N(\mathfrak{a}) \mid M : R\text{-module}, N \subseteq M\},$$
$$c^J(\mathfrak{a}) \mapsto c_{R/J}^{R/J}(\mathfrak{a}).$$

— Answer of Q. 2, Q. 3 —

$$c^i \leq c^{\tau(\mathfrak{a}^{c^i})}(\mathfrak{a}).$$
Moreover, if R: a $\mathbb{Q}$-Goren. normal local.
$$\{c^i(\mathfrak{a}) \mid i \in \mathbb{Z}_{>0}\} \hookrightarrow \{c_E^N(\mathfrak{a}) \mid N \subseteq E\},$$
$$c^i \mapsto c_E^N(\mathfrak{a}),\ N := 0_E^{*\mathfrak{a}^{c^i}}.$$

3 On toric rings

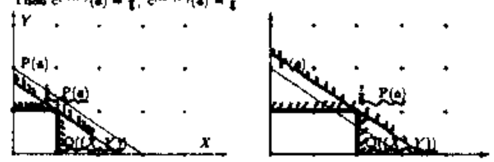
— Note. and Assum. —

- $N = \mathbb{Z}^d$, $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$, $M_{\mathbb{R}} := M \otimes \mathbb{R}$
  $\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$
  $\sigma := \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n$: a str. conv. rat. poly. cone of $N_{\mathbb{R}}$, $\{v_j\}$: primitive
  $\sigma^\vee := \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, \forall v \in \sigma\}$
- $R = k[\sigma^\vee \cap M]$: a toric ring, k: a perfect field
  $\mathfrak{a} = (X^{u_1}, \ldots, X^{u_s})$: a monomial ideal of R
- $\mathrm{O} := \{\omega \in \sigma^\vee \mid \forall j, \langle \omega, v_j \rangle < 1\} \subseteq \sigma^\vee$.
  $\mathrm{P}(\mathfrak{a}) := \operatorname{Conv}\{u \in \sigma^\vee \cap M \mid X^u \in \mathfrak{a}\} \subseteq \sigma^\vee$
  $\mathrm{Q}(\mathfrak{a}) := \bigcup_{X^u \in \mathfrak{a}} u + \sigma^\vee \subseteq \sigma^\vee$.

The following is the main theorem of this poster. Watanabe suggested the idea of the theorem.

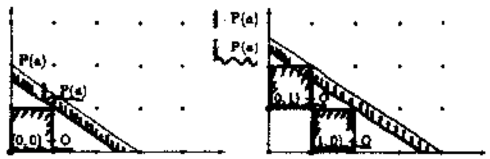
Theorem 9 (Main Theorem).
$$c^J(\mathfrak{a}) = \sup_{\omega \in \sigma^\vee \setminus \mathrm{Q}(J)} \lambda(\omega),$$
where $\lambda(\omega) := \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid \omega \in \lambda \cdot \mathrm{P}(\mathfrak{a})\}$

Example. $R = k[X, Y]$, $\mathfrak{a} = (X^3, Y^2)$.
Then $c^{(X,Y)}(\mathfrak{a}) = \frac{5}{6}$, $c^{(X^2,Y)}(\mathfrak{a}) = \frac{7}{6}$



Theorem 10 (Howald type [HY03], [B04]).
$$\tau(\mathfrak{a}^t) \cong (X^u \mid (u + \mathrm{O}) \cap (t \cdot \mathrm{P}(\mathfrak{a})) \neq \emptyset)$$

Example. $R = k[X, Y]$, $\mathfrak{a} = (X^3, Y^2)$. Then,
$$\tau(\mathfrak{a}^t) = \begin{cases} R, & t < \frac{5}{6}, \\ (X, Y), & \frac{5}{6} \leq t < \frac{7}{6}, \\ (X^2, Y), & \frac{7}{6} \leq t < \frac{4}{3} \end{cases}$$



This implies a formula of F-jump. coeff.

Corollary 11.
$$c^1(\mathfrak{a}) = \sup_{\omega \in \mathrm{O}} \lambda(\omega) = \inf_{X^u \in R} \mu(u),$$
where $\mu(u) := \sup_{\omega \in \mathrm{O}} \lambda(u + \omega)$. In general, $i \in \mathbb{Z}_{>0}$,
$$c^i(\mathfrak{a}) = \inf_{X^u \notin \tau(\mathfrak{a}^{c^{i-1}})} \mu(u).$$

— Answer of Q. 2 —

Corollary 12. R: a toric ring, $\mathfrak{a}$: a monomial ideal
$$c^i \leq c^{\tau(\mathfrak{a}^{c^i})}(\mathfrak{a}),\ \forall i.$$

Sketch of pf. $\exists X^u \in \tau(\mathfrak{a}^{c^{i-1}}) \setminus \tau(\mathfrak{a}^{c^i})$. $u + \mathrm{O} \subseteq \sigma^\vee \setminus \mathrm{Q}(\tau(\mathfrak{a}^{c^i}))$.
Then, $c^i \leq \sup_{\omega' \in u + \mathrm{O}} \lambda(\omega') \leq \sup_{\omega' \in \sigma^\vee \setminus \mathrm{Q}(\tau(\mathfrak{a}^{c^i}))} \lambda(\omega') = c^{\tau(\mathfrak{a}^{c^i})}(\mathfrak{a})$. □

4 Applications

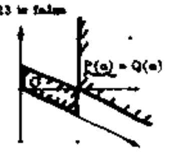
— Part of answer of Q. 3 —

Proposition 13. R: an r-Goren. toric ring ($r > 1$). Then,
$$c^1(\mathfrak{a}) < c^{\tau(\mathfrak{a}^{c^1})}(\mathfrak{a}),\ \forall \mathfrak{a}: \text{monomial ideals}.$$

Sketch of pf. $\exists \omega \in \sigma^\vee$ s. t. $\langle \omega, v_j \rangle = 1$, $\forall j$ and $\omega \notin M$.
Then, $c^1(\mathfrak{a}) = \lambda(\omega) < (1 + \varepsilon)\lambda(\omega) \leq c^{\tau(\mathfrak{a}^{c^1})}(\mathfrak{a})$. □

Example. If R: Goren. Sing., then Prop 13 is false.

$R = k[X, Y, X^2Y^{-1}]$: A_1-Sing.,
$\mathfrak{a} = (X)$. Then,
$c^1(\mathfrak{a}) = 1$, $\tau(\mathfrak{a}^1) = (X)$, $c^{(X)}(\mathfrak{a}) = 1$

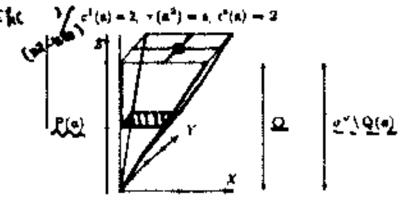


— Part of answer of Q. 3 —

Proposition 14. R: a toric ring defined by a simplicial cone.

$$\exists \mathfrak{a}: \mathfrak{m}\text{-primary ideal s. t. } c^1(\mathfrak{a}) = c^{\tau(\mathfrak{a}^{c^1})}(\mathfrak{a}) \Rightarrow R: \text{regular ring}$$

Example. If R defined by non-simplicial cone, then Prop. 14 is false.
$R = k[XZ, YZ, XYZ, Z]$ [redacted], $\mathfrak{a} = (XZ, YZ, XYZ, Z)$. Then,
$c^1(\mathfrak{a}) = 2$, $\tau(\mathfrak{a}^2) = \mathfrak{a}$, $c^{\mathfrak{a}}(\mathfrak{a}) = 2$



Remark. Known results about rationality of F-thresholds and F-jump. coeff.

[BMS06] R: a regular ring of essentially of finite type of k
$\Rightarrow \forall \mathfrak{a}$, $c^J(\mathfrak{a})$, $c^i(\mathfrak{a}) \in \mathbb{Q}$

[BMS07] R: a regular ring
$\Rightarrow \forall f \in R$, $c^J((f))$, $c^i((f)) \in \mathbb{Q}$

— Part of Answer of Q. 1 —

Proposition 15. R: a toric ring defined by a simplicial cone,
$\mathfrak{a}$: a monomial ideal,
J: an $\mathfrak{m}$-primary monomial ideal.
$\Rightarrow c^J(\mathfrak{a}) \in \mathbb{Q}$

Sketch of pf. J: $\mathfrak{m}$-primary.
$\Rightarrow \sigma^\vee \setminus \mathrm{Q}(J)$: bounded,
$\Rightarrow \partial\mathrm{Q}(J)$: bounded.
In general,
$$\omega, \omega' \in \sigma^\vee \Rightarrow \lambda(\omega) \leq \lambda(\omega + \omega').$$
Then,
$$c^J(\mathfrak{a}) = \max_{u \in \partial\mathrm{Q}(J)} \lambda(u)$$
$\sigma^\vee$: simplical $\Rightarrow$ the following set is a finite set of rational pt. as