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On codim4 $\mathbb{Q}$-Fano 3 Folds with Fano index 2

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1 Main Results

We study Fano 3-folds with Fano index 2: that is, 3-folds X with Picard rank 1, $\mathbb{Q}$-factorial terminal singularities and $-K_X = 2A$ for an ample Weil divisor A. We gave a first classification of all possible Hilbert series of such polarised varieties X, A and list 33 families that can be realised in codimension up to 4. See, Brown-Suzuki, *Fano 3-folds with divisible anticanonical class*, *Manuscripta Math.*123, 2007, pp.37–51.

We work over the complex number field $\mathbb{C}$.

2 The classification of $\mathbb{Q}$-Fano 3-folds

The *Fano index* $f = f(X)$ of a Fano 3-fold X is the largest positive integer such that $-K_X = fA$ for some Weil divisor A. Equality of divisors denotes linear equivalence of some multiple. A Weil divisor A is called a *primitive ample divisor*.

History

1. Smooth — Iskovskikh (1980)
2. Toric — Borisov-Borisov (1993)
3. $f = 1$
   MMP $\longrightarrow$ Mori, Mukai, Takagi (many)
   Graded Ring Method $\longrightarrow$ Altinok, Brown, Corti, Fletcher, Reid
4. $f \geq 2$ and codimension up to 3
   Graded Ring Method $\longrightarrow$ Suzuki [Su1]

3 The graded ring method

A Fano 3-fold X with primitive ample divisor A has a graded ring $R(X, A) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nA))$. This graded ring is finitely generated. The *Hilbert series* of X, A is $P(X, A) = \sum_{n=0}^{\infty} \chi(\theta_X(nA))t^n$. Each $\chi(\theta_X(nA))$ is given by the singular Riemann-Roch formula ([YPG, Su2]) and then

Thm 1 ([Su2])

$$P_{X,A}(t) = \frac{1}{1-t} + \frac{t}{(1-t)^4}A^3 + \frac{t}{(1-t)^2}\frac{Ac_2(X)}{12} + \sum_{p \in \mathcal{B}} c_p(t)$$

where, $c_p(t)$ is equal to

$$\frac{1}{1-t^r}\left(\sum_{k=1}^{r-1}\left(\frac{-i_k(r^2-1)}{12r} + \sum_{j=1}^{i_k-1}\frac{\overline{bj}(r-\overline{bj})}{2r}\right)t^k\right).$$

1. **$\bar{x}$: the smallest residue mod r**
2. **$\mathcal{B} := \left\{\frac{1}{r_n}(a_n, r_n - a_n, 2)\right\}$ $(n = 1, 2, 3 \ldots)$.**
3. **i: the smallest positive integer which satisfies $nA \sim_{\mathbb{Q}} iK_X$.**

A choice of minimal homogeneous generating set $x_0, \ldots, x_N \in R(X, A)$ determines an embedding $X \hookrightarrow \mathbb{P}^N = \mathbb{P}(\alpha_0, \ldots, \alpha_N)$ for some weighted projective space $\mathbb{P}^N$, where $x_i \in H^0(X, \mathcal{O}_X(\alpha_i A))$. With this embedding, we say that X has *codimension* $N - 3$.

Ex. 1 **Codimension 1 hypersurface case**

$$\mathcal{B} = \left\{\frac{1}{3}(1,2,2), \frac{1}{5}(1,4,2), \frac{1}{11}(3,8,2)\right\}$$

$$P(X,t) = \frac{1-t^{38}}{(1-t^2)(1-t^3)(1-t^5)(1-t^{11})(1-t^{19})}$$

$$\Longrightarrow X_{38} \subset \mathbb{P}(2,3,5,11,19)$$

It is known that the embedding of codimension 2 [resp.3] is defined by the complete intersection [resp. five maximal Pfaffians of a 5× 5 skew matrix]. 36 can be realized by (X, A) of codimension up to 3.

Next we consider the embedding of codimension 4. If $|-K_X|$ contains a K3 surface S, we may compare with those of [B] to the constructions we might make.

Consider $X \subset \mathbb{P}(2,2,3,5,5,7,12,17)$ with $\mathcal{B} = \{\frac{1}{17}(5,12,2)\}$ as an example.

Ex.2 **Codimension 4 case**

$S \subset \mathbb{P}(2,3,5,5,7,12,17)$
↓ **Type I projection from** $1/17(5,12)$
$T_{10,12,14,15,17} \subset \mathbb{P}(2,3,5,5,7,12)$
↓ **Type I projection from** $1/12(5,7)$
$Z_{10,12} \subset \mathbb{P}(2,2,3,5,5,7)$

Using Type II unprojection, we construct from the example above

$$\mathbb{P}(2,5,12) \subset Y_{10,12,14,15,17} \subset \mathbb{P}(2,2,3,5,5,7,12)$$

as a linear subspace in the codimension 3 weak Fano 3-fold. By projection and unprojection again, we have $\mathbb{Q}$-Fano 3-fold in codimension 4.

4 Database

We use computer programs by Magma at the final stage.

If there is a $\mathbb{Q}$-Fano 3-fold X of Fano index 2, there must be integer solutions r_k, a_k of the following equations and inequalities:

Conditions

1. **$\sum_{k=1}^{m}(r_k - \frac{1}{r_k}) < 24$, $(r_k, a_k) = 1$ and $(2, r_k) = 1$ for all k**
2. **$0 \leq A^3 \leq 4/5AC_2$ (Kawamata Boundedness)**
3. **$\chi(\mathcal{O}_X(-A)) = 0$**

Then we have whole possible candidates of $\mathcal{B}$ and their Hilbert series. As a result, we have the following list of $\mathbb{Q}$-Fano 3-folds in codimension 4.

Remark **Lists of examples available at http://malham.kent.ac.uk/grdb/FanoForm.php**

Question

The existence problem of two remaining candidates

1. **$\mathcal{B} = \{5 \times 1/3(1,2,2), 1/5(1,4,2)\}$, $A^3 = 1/15$**
2. **$\mathcal{B} = \{3 \times 1/3(1,2,2), 1/5(2,3,2), 1/7(1,6,2)\}$, $A^3 = 1/35$**

is still open, but both are expected not to exist.

Ambient $\mathbb{P}^7$	Basket $\mathcal{B}$	A^3
$\mathbb{P}(1,1,1,1,1,1,1,1)$	nonsingular	6
$\mathbb{P}(1,1,1,1,1,2,2,3)$	$\frac{1}{3}$	10/3
$\mathbb{P}(1,1,1,2,2,2,3,3)$	$2 \times \frac{1}{3}$	5/3
$\mathbb{P}(1,1,1,2,2,2,3,5)$	$\frac{1}{5}(2,3,2)$	8/5
$\mathbb{P}(1,1,1,2,2,3,4,5)$	$\frac{1}{5}(1,4,2)$	7/5
$\mathbb{P}(1,1,2,2,2,3,3,3)$	$3 \times \frac{1}{3}$	1
$\mathbb{P}(1,1,2,2,2,3,3,5)$	$\frac{1}{3}, \frac{1}{5}(2,3,2)$	14/15
$\mathbb{P}(1,1,2,2,2,3,5,7)$	$\frac{1}{7}(2,5,2)$	6/7
$\mathbb{P}(1,1,2,2,3,3,4,5)$	$\frac{1}{3}, \frac{1}{5}(1,4,2)$	11/15
$\mathbb{P}(1,1,2,2,3,3,4,7)$	$\frac{1}{7}(3,4,2)$	5/7
$\mathbb{P}(1,1,2,3,4,5,6,7)$	$\frac{1}{7}(1,6,2)$	3/7
$\mathbb{P}(1,2,2,3,3,3,4,5)$	$3 \times \frac{1}{3}, \frac{1}{5}(1,4,2)$	2/5
$\mathbb{P}(1,2,2,3,3,3,4,7)$	$2 \times \frac{1}{3}, \frac{1}{7}(3,4,2)$	8/21
$\mathbb{P}(1,2,2,3,3,4,5,5)$	$\frac{1}{3}, \frac{1}{5}(1,4,2), \frac{1}{5}(2,3,2)$	1/3
$\mathbb{P}(1,2,2,3,3,4,5,7)$	$\frac{1}{5}(2,3,2), \frac{1}{7}(3,4,2)$	11/35
$\mathbb{P}(1,2,2,3,3,5,8,11)$	$\frac{1}{11}(3,8,2)$	3/11
$\mathbb{P}(1,2,2,3,4,5,5,7)$	$\frac{1}{5}(1,4,2), \frac{1}{7}(2,5,2)$	9/35
$\mathbb{P}(1,2,3,4,4,5,5,5)$	$3 \times \frac{1}{5}(1,4,2)$	1/5
$\mathbb{P}(1,2,3,4,4,5,5,9)$	$\frac{1}{5}(1,4,2), \frac{1}{9}(4,5,2)$	8/45
$\mathbb{P}(1,2,3,4,4,5,9,13)$	$\frac{1}{13}(4,9,2)$	2/13
$\mathbb{P}(1,2,3,4,5,5,6,7)$	$\frac{1}{3}, \frac{1}{5}(1,4,2), \frac{1}{7}(1,6,2)$	17/105
$\mathbb{P}(1,2,3,4,5,5,6,11)$	$\frac{1}{3}, \frac{1}{11}(5,6,2)$	5/33
$\mathbb{P}(1,2,3,4,5,6,7,7)$	$\frac{1}{7}(1,6,2), \frac{1}{7}(3,4,2)$	1/7
$\mathbb{P}(1,2,3,5,6,7,8,9)$	$2 \times \frac{1}{3}, \frac{1}{9}(1,10,2)$	1/9
$\mathbb{P}(1,2,5,7,8,9,10,11)$	$\frac{1}{5}(2,3,2), \frac{1}{11}(1,10,2)$	3/5
$\mathbb{P}(2,2,3,3,4,5,5,5)$	$\frac{1}{3}, 3 \times \frac{1}{5}(2,3,2)$	2/15
$\mathbb{P}(2,2,3,3,4,5,5,7)$	$2 \times \frac{1}{3}, \frac{1}{5}(2,3,2), \frac{1}{7}(2,5,2)$	13/105
$\mathbb{P}(2,2,3,3,4,5,7,9)$	$3 \times \frac{1}{3}, \frac{1}{9}(2,7,2)$	1/9
$\mathbb{P}(2,2,3,5,5,7,12,17)$	$\frac{1}{17}(5,12,2)$	1/17
$\mathbb{P}(2,3,3,4,5,5,6,7)$	$5 \times \frac{1}{3}, \frac{1}{5}(1,4,2)$	1/15
$\mathbb{P}(2,3,3,4,5,7,10,13)$	$2 \times \frac{1}{3}, \frac{1}{13}(3,10,2)$	
$\mathbb{P}(2,3,4,5,5,6,7,7)$	$\frac{1}{3}, \frac{1}{5}(1,4,2), \frac{1}{5}(2,3,2), \frac{1}{7}(3,4,2)$	1/21
$\mathbb{P}(2,3,4,5,5,6,7,9)$	$2 \times \frac{1}{3}, \frac{1}{5}(2,3,2), \frac{1}{9}(4,5,2)$	2/45
$\mathbb{P}(2,3,5,6,7,7,8,9)$	$3 \times \frac{1}{3}, \frac{1}{5}(2,3,2), \frac{1}{7}(1,6,2)$	1/35
$\mathbb{P}(2,5,5,6,7,8,9,11)$	$2 \times \frac{1}{5}(2,3,2), \frac{1}{11}(5,6,2)$	1/55

Table 1: $\mathbb{Q}$-Fano 3-folds in codimension 4

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