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Stability of direct images of cotangent bundles by Frobenius morphisms

joint work with Hideyasu Sumihiro (Hiroshima University)
Yukinori KITADAI (Hiroshima University)

Introduction

$k = \bar{k}$: a field, char$(k) = p > 0$
X: nonsing. proj. var.$/k$, $\dim X = n$
$F = F_X$: the absolute Frobenius morphism of X

Theorem (H. Lange-C. Pauly (2003)).
X: nonsing. proj. curve$/k$, $g(X) \geq 2$
$\mathscr{L}$: line bundle on X
$\Longrightarrow F_*\mathscr{L}$: stable

a natural generalization

Problem.
X: nonsing. proj. var. of general type$/k$
H: numerically positive divisor on X
$\mathscr{E}$: semistable vector bundle on X w.r.t. H
$\Longrightarrow F_*\mathscr{E}$ is semistable w.r.t. H?

Main tools

Canonical filtrations

A useful filtration of $F^*F_*\mathscr{O}_X$
$\varphi : F^*F_*\mathscr{O}_X \to \mathscr{O}_X$: natural surjection
$F^*F_*\mathscr{O}_X$: $\mathscr{O}_X$-algebra
$I := \ker\varphi$
So we get
$F^*F_*\mathscr{O}_X \supset I \supset I^2 \supset I^3 \supset \cdots$.
$I^\bullet$: canonical filtration of $F^*F_*\mathscr{O}_X$
$\mathscr{E}$: a vector bundle on X
$W^i := F^*F_*\mathscr{E} \cdot I^i$
$W^\bullet$: canonical filtration of $F^*F_*\mathscr{E}$

Canonical connections
$\nabla : F^*F_*\mathscr{O}_X \to F^*F_*\mathscr{O}_X \otimes \Omega^1_X$

Results

Theorem (K-Sumihiro (2006)).
X: nonsing. proj. surf.$/k$
H: num. pos. div.$/X$ s.t.
$|mH|$: base pt free
and contains nonsing. member $(m \gg 0)$
$K_XH > 0$, Assume Ω^1_X: semistable w.r.t. H
$\Longrightarrow F_*\mathscr{L}$ is semistable w.r.t. H
for any line bundle $\mathscr{L}$ on X.

Higher dimensional case (surface)

Theorem (K-Sumihiro (2006), Mehta-Pauly, Sun).
X: nonsing. proj. curve$/k$, $g(X) \geq 2$
$\mathscr{E}$: semistable (resp. stable) vector bundle on X
$\Longrightarrow F_*\mathscr{E}$ is semistable (resp. stable).

Higher rank case

Main Theorem

(K-Sumihiro (2007))
X: nonsing. proj. surf.$/k$
H: num. pos. div.$/X$ s.t.
$|mH|$: base pt free
and contains nonsing. member $(m \gg 0)$
$K_XH > 0$, Assume Ω^1_X: semistable w.r.t. H
$\Longrightarrow F_*(\Omega^1_X \otimes \mathscr{L})$ is semistable w.r.t. H
for any line bundle $\mathscr{L}$ on X.

Higher dimensional and higher rank case (surface, cotangent bundle)

An application

An application for the geography of nonsing. proj. minimal surf. of gen. type

X: nonsing. proj. minimal surf. of gen. type$/k$
Assume Ω^1_X: semistable w.r.t. K_X

(1) (Bogomolov's inequality)
Ω^1_X: strongly semistable
($\Longleftrightarrow (F^k)^*\,\Omega^1_X$: semistable $\forall k$) w.r.t. K_X,
$\Longrightarrow c_1^2(X) \leq 4c_2(X)$.

(2) $(F^{k-1})^*\Omega^1_X$: semistable w.r.t. K_X
and $(F^k)^*\Omega^1_X$: not semistable w.r.t. K_X $\exists k$,
$$\Longrightarrow c_1^2(X) \leq \frac{4p^{2k}}{p^{2k} - (p-1)^2}c_2(X).$$
In particular, $c_2(X) > 0$.