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Division polynomials and canonical local heights on hyperelliptic Jacobians

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1 Introduction

In the theory of elliptic curves, the division polynomials are important for the study of the structure of the torsion subgroup. Moreover the division polynomials have various applications. For example, they appear in some equalities of the canonical local height functions.

The author generalized these equalities to the Jacobian varieties of curves of genus two, and reported them in this symposium last year. In this poster, we generalize them to the Jacobian varieties of general hyperelliptic curves over $\mathbb{C}$.

2 Review on the theory of hyperelliptic functions

Let C be a non-singular projective curve of genus g over $\mathbb{C}$ defined by

$$y^2 = f(x) = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_1 x + \lambda_0.$$

Then $f(x)$ has no multiple roots and C has the unique point ∞ at infinity.

For $j = 1, 2, \ldots, g$, let

$$\omega_j = \frac{x^{j-1}dx}{2y}, \quad \eta_j = \frac{1}{2y}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+j}x^k dx.$$

Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be a symplectic basis of $H_1(C, \mathbb{Z})$. Then their intersections satisfy $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$, where δ_{ij} is Kronecker's delta. We define the period matrices by

$$\omega' = \left(\int_{\alpha_j} \omega_i\right), \omega'' = \left(\int_{\beta_j} \omega_i\right), \eta' = \left(-\int_{\alpha_j} \eta_i\right), \eta'' = \left(-\int_{\beta_j} \eta_i\right).$$

We write $\boldsymbol{e}(z) = \exp(2\pi\sqrt{-1}\,z)$. We define the theta function with characteristics by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{n \in \mathbb{Z}^g} \boldsymbol{e}\left(\frac{1}{2}{}^t(n+a)\tau(n+a) + {}^t(n+a)(z+b)\right),$$

where $z \in \mathbb{C}^g$, $\tau \in M_g(\mathbb{C})$ is symmetric, $\operatorname{Im}(\tau)$ is positive definite, and $a, b \in \mathbb{R}^g$. Let $\Lambda = \omega'\mathbb{Z}^g + \omega''\mathbb{Z}^g$. Then Λ is a lattice of $\mathbb{C}^g$. Let

$$\delta'' = {}^t\left(\frac{1}{2}, \ldots, \frac{1}{2}\right), \quad \delta' = {}^t\left(\frac{g}{2}, \frac{g-1}{2}, \ldots, \frac{1}{2}\right), \quad \delta = \begin{pmatrix} \delta'' \\ \delta' \end{pmatrix}.$$

Definition 1. We define the *hyperelliptic sigma function* on $\mathbb{C}^g$ by

$$\sigma(u) = c\exp\left(\frac{1}{2}{}^t u\eta'\omega'^{-1}u\right)\vartheta[\delta](\omega'^{-1}u),$$

where c is a certain constant.

For any $u \in \mathbb{C}^g$, we denote by u' and u'' the elements in $\mathbb{R}^g$ satisfying $u = \omega'u' + \omega''u''$. We define a $\mathbb{C}$-valued $\mathbb{R}$-bilinear form on $\mathbb{C}^g \times \mathbb{C}^g$ by

$$L(u, v) = \frac{1}{2\pi\sqrt{-1}}{}^t u(\eta'v' + \eta''v'').$$

For $l \in \Lambda$, let

$$\chi(l) = \boldsymbol{e}\left(\left({}^t l'\delta'' - {}^t l''\delta'\right) + \frac{1}{2}{}^t l'l''\right).$$

Note that $\chi(l) = \pm 1$.

Proposition 2 (Translational relation)**.** *For any $u \in \mathbb{C}^g$ and any $l \in \Lambda$, we have*

$$\sigma(u+l) = \chi(l)\boldsymbol{e}\left(L\left(u + \frac{1}{2}l, l\right)\right)\sigma(u).$$

We define the hyperelliptic $\wp$-functions by

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j}\log\sigma(u),$$
$$\wp_{ijk}(u) = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k}\log\sigma(u), \ \ldots, \quad i, j, k, \ldots \in \{1, 2, \ldots, g\}.$$

Then the hyperelliptic $\wp$-functions and their derivatives are periodic with respect to Λ.

Let J be the Jacobian variety of C. We consider $J = \mathbb{C}^g/\Lambda$ and let $p\colon \mathbb{C}^g \to J$ be the natural projection. We can consider $\wp_{ij}$, $\wp_{ijk}$, $\ldots$ as meromorphic functions on J.

We define the theta divisor Θ by

$$\Theta = p(\{u \in \mathbb{C}^g \mid \sigma(u) = 0\}).$$

We have the addition formula as follows:

Theorem 3 (Buchstaber-Enolskii-Leykin)**.** *We have*

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \mathcal{F}_g(u, v),$$

where $\mathcal{F}_g(u, v)$ is a polynomial in $\wp_{ij}(u)$ and $\wp_{ij}(v)$ with integral coefficients.

3 The division polynomials

Definition 4. Let n be an integer. We define the *division polynomial* ϕ_n by

$$\phi_n(u) = \frac{\sigma(nu)}{\sigma(u)^{n^2}}.$$

Then $\phi_n(u)$ is periodic with respect to Λ. Therefore it is a meromorphic function on J. Since $\phi_n(u)$ has no poles except along Θ, it is represented as a polynomial in $\wp_{ij}(u)$ and $\wp_{ijk}(u)$. More precisely, the following theorems hold.

Theorem 5. *We have*

$$\phi_0(u) = 0, \quad \phi_1(u) = 1, \quad \phi_2(u) = \left.\frac{\partial^l \mathcal{F}_g(v, u)}{\partial v_1 \partial v_3 \ldots \partial v_{2l-1}}\right|_{v=u},$$

*where $g = 2l - 1$ or $g = 2l$. In particular, $\phi_0(u)$, $\phi_1(u)$, and $\phi_2(u)$ are polynomials in $\wp_{ij}(u)$ and $\wp_{ijk}(u)$ with coefficients in the **ring** $\mathbb{Q}[\lambda_i]$.*

Theorem 6. *For any integer n, $\phi_n(u)$ is a polynomial in $\wp_{ij}(u)$ and $\wp_{ijk}(u)$ with coefficients in the **field** $\mathbb{Q}(\lambda_i)$.*

We conjecture a stronger statement.

Conjecture 7. *For any integer n, $\phi_n(u)$ is a polynomial in $\wp_{ij}(u)$ and $\wp_{ijk}(u)$ with coefficients in the **ring** $\mathbb{Q}[\lambda_i]$.*

Remark. This conjecture is true for $g = 1, 2$. The case where $g = 1$ is well-known, and the case where $g = 2$ is described by Kanayama (2005).

4 The canonical local height function

We give an explicit formula for the canonical height function (Néron function) on the Jacobian J over $\mathbb{C}$. We begin with the following definition.

Definition 8. The function $\hat{\lambda}\colon J \setminus \Theta \to \mathbb{R}$ is called the *canonical local height function* (associated with Θ) if the following conditions are satisfied.

(i) $\hat{\lambda}$ is a local height function (Weil function) associated with Θ.

(ii) For all $P \in J \setminus \Theta$,

$$\hat{\lambda}([2]P) = 4\hat{\lambda}(P) - \log|\phi_2(P)|.$$

Note that $\hat{\lambda}$ is uniquely determined.

We have the following theorem and corollaries.

Theorem 9. *Let $P \in J\setminus\Theta$. We can take $u \in \mathbb{C}^g$ such that $p(u) = P$. Then we have*

$$\hat{\lambda}(P) = -\log\left|\boldsymbol{e}\left(-\frac{1}{2}L(u, u)\right)\sigma(u)\right|.$$

Corollary 10 (Quasi-parallelogram law)**.** *Let $P, Q \in J$. If $P, Q, P+Q, P-Q \notin \Theta$, then we have*

$$\hat{\lambda}(P+Q) + \hat{\lambda}(P-Q) = 2\hat{\lambda}(P) + 2\hat{\lambda}(Q) - \log|\mathcal{F}_g(P, Q)|.$$

Corollary 11. *Let n be a non-zero integer and $P \in J \setminus \Theta$. If $[n]P \notin \Theta$, then we have*

$$\hat{\lambda}([n]P) = n^2\hat{\lambda}(P) - \log|\phi_n(P)|.$$