代数幾何学シンポジウム記録
2007年度 pp.107-112

ALGEBRAIC SPACES AND SCHEMES

KAZUHIRO FUJIWARA

ABSTRACT. In this paper, a relation between formal algebraic paces and formal schemes is discussed. In particular, we show that, for a formal algebraic space, there is a modification which is a formal scheme (equivalence theorem) under a mild assumption.

1. INTRODUCTION

Algebraic spaces were first introduced by M. Artin in around 1970. An algebraic space X is written as a quotient S/R as a sheaf on the large étale site of the schemes, where S and R are schemes, and R is an étale equivalence relation, i.e., each projection $R \to S \times S \stackrel{\mathrm{pr}_i}{\to} S$ is étale.

Though it is a rather technical object, it appears naturally in algebraic geometry, in particular as the coarse moduli spaces of various moduli spaces.

Over $\mathbb{C}$, a GAGA-type functor

$$\{ \text{ Algebraic spaces } /\mathbb{C}\} \stackrel{\mathrm{an}}{\to} \{ \text{ Complex analytic spaces } \}$$

is defined by taking the quotient $S(\mathbb{C})/R(\mathbb{C})$ in the category of complex analytic spaces. Artin showed that this functor induces a categorical equivalence

$$\{ \text{ Proper algebraic spaces } /\mathbb{C}\} \stackrel{\mathrm{an}}{\simeq} \{ \text{ Moishezon spaces } \}.$$

One of the purpose of this article is to consider a non-archimedian analogue of the GAGA-functor.

Let us recall the non-archimedian analogue of the complex analytic spaces briefly. We fix a pair (V, a) of a valuation ring V and a non-zero element $a \in V$. We assume V is complete for the a-adic topology, and denote the fraction field by K. As an analogue of the complex analytic spaces, Tate defined the notion of rigid analytic spaces over K [8]. After Tate, Raynaud made clear the relationship between rigid analytic spaces over K and formal schemes over V, and any formal scheme X of finite type over V defines a rigid analytic space $(X)^{\mathrm{rig}}$, which is viewed as a birational equivalence class of X modulo admissible blow ups in the category of formal schemes [6].

Then we have the following theorem (joint work with F. Kato):

Theorem 1.1. *There is a GAGA functor*

$$\{ \textit{ Separated algebraic spaces } /K\}$$
$$\stackrel{\mathrm{an}}{\to} \{ \textit{ Rigid analytic spaces } /K\}$$

with obvious properties (commutes with fiber products, etc.).

Let us sketch the idea of the proof of theorem 1.1 briefly. The details will appear in a book in preparation [4] (cf. also our survey article [3]).

Using the Nagata embedding theorem for algebraic spaces, it suffices to construct the GAGA functor for proper algebraic spaces over K (we hope to include a proof of embedding theorem as well in [4]). For a proper algebraic space Y over K, take a proper model Y' of Y

The author is partially supported by JSPS grant. The grant number is 17340002.

over V as an algebraic space using the Nagata embedding theorem, then set $X = \hat{\bar{Y}}$. Then we have a formal scheme X' and a formal modification $X' \to X$ by the following theorem:

Theorem 1.2. *(Equivalence theorem) Let X be a formal algebraic space of finite type over V. Then there is an admissible blow up $X' \to X$ such that X' is a formal scheme.*

Since X' is a formal scheme over V, one has the generic fiber $(\widehat{X'})^{\mathrm{rig}}$ as a rigid analytic space of Tate-Raynaud. This is independent of any choices of X and X', and is the desired space Y^{an}.

Remark 1.3. (1) *The equivalence theorem is shown by using Nagata's method of using Zariski's Riemann spaces.*
(2) *The existence of a GAGA-functor is also shown by Conrad-Temkin* [2] *(the existence of rigid-analytic quotients by étale equivalence relations).*

In the rest of the article, we discuss the main ingredients of the proof of the equivalence theorem.

2. Formal schemes and their Visualization

For simplicity, we restrict our consideration to the noetherian case, and fix a noetherian formal scheme S as a base space in the sequel. For basic properties of formal schemes in the noetherian case, see [EGA]. The treatment of formal schemes including both noetherian and valuation rings will be offered in [4].

When X is a formal scheme of finite type over S, X is regarded as an inductive limit

$$X = \varinjlim_n X_n$$

in the category of topologically local ringed spaces, where

$$X_n = \operatorname{Spec} \mathscr{O}_X / I^{n+1},$$

I is an ideal of definition of X.

Let X' be an admissible blow up of X, i.e,

$$X' \simeq Bl(J)^{\wedge},$$

$$Bl(J) = \operatorname{Proj} \oplus_{n \geq 0} J^n,$$

where J is a coherent ideal which contains I^N for some N (i.e., open for the I-adic topology). The category B_X of all admissible blowing ups of X is naturally filtered.

Then the Zariski-Riemann space $ZR(X)$ of X is defined as

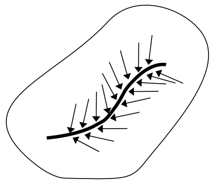
$$ZR(X) = \varprojlim_{\pi : X' \to X, \pi \in B_X} X'$$

in the category of local ringed spaces.

By the definition, a point of $ZR(X)$ is described as a compatible system $\{x_{X'}\}_{X' \in BL_X}$, where $x_{X'}$ is a point of X' as a topological space.

Note that any morphism $\operatorname{Spf} V \to X$, where V is a valuation ring which is complete for IV-adic topology, defines a point of $ZR(X)$ in the following way: By the valuative criterion for properness, $\operatorname{Spf} V \to X$ is factored into $\operatorname{Spf} V \to X' \to X$ for any admissible blow up X' of X, and hence the images of the closed point of $\operatorname{Spf} V$ forms a compatible system and is a point of $ZR(X)$.

It is verified that any point of $ZR(X)$ is obtained from some valuation ring V and a morphism $\operatorname{Spf} V \to X$ in this way, so $ZR(X)$ is regarded as a space of arcs since $\operatorname{Spf} V$ is an analogue of an arc.



The following theorem is the formal scheme analogue of Zariski's theorem for algebraic function fields, which plays a central role in the following:

Theorem 2.1. *When X is coherent, i.e., quasi-compact and quasi-separated, $ZR(X)$ is quasi-compact and quasi-separated as a topological space.*

3. Variant for formal algebraic spaces

We generalize the visualization construction for formal algebraic spaces. First we recall basic properties of algebraic spaces:

- (Knudson [5]) Let X be a coherent (=quasi-compact and quasi-separated) algebraic space. Then the following conditions are equivalent:
  (1) The global section functor from the category of quasi-coherent sheaves
  $$\Gamma : Qcoh_X \to (Ab)$$
  is exact.
  (2) X is an affine scheme.
- Let X be a coherent algebraic space, X_0 a closed subspace defined by a quasi-coherent ideal sheaf J which is nilpotent ($J^s = 0$ for some $s \geq 1$). If X_0 is a scheme, then X is also a scheme.
- (The existence of a stratification by schemes, [5], [7]) Let X be a coherent algebraic space. Then there is a stratification $Z_0 = X_0 \supset Z_1 \supset \cdots Z_N \supset Z_{N+1} = \emptyset$ where Z_α is finitely presented closed subspace, and each strata $W_\alpha = Z_\alpha \setminus Z_{\alpha+1}$ is a scheme.
- (Chow's lemma) Let S be a scheme, X be an algebraic space which is separated and finitely presented over S. For a quasi-compact open subspace $U \subset X$ which is a scheme, there is an admissible blow up $X' \to X$ centered in $X \setminus U$ such that X' is a scheme. If U is quasi-projective over S, one may take X' to be quasi-projective.
- (Limit theorem, [5], [7]) Let X be a coherent algebraic space. Then any quasi-coherent sheaf on X is the inductive limit of sub quasi-coherent sheaves of finite type.
- (Extension theorem, [5], [7])) Let X be a coherent algebraic space, $j : U \hookrightarrow X$ be a quasi-compact open immersion. For any quasi-coherent sheaf $\mathscr{F}$ of finite type on U, there is a quasi-coherent sheaf $\mathscr{G}$ of finite type on X which extends $\mathscr{F}$, i.e., $\mathscr{G}|_U = \mathscr{F}$.

The notion of formal algebraic spaces is defined as in the definition of formal schemes, using algebraic spaces instead of schemes.

As in §2, we consider the noetherian case for simplicity, and fix a noetherian formal scheme S as a base space. Let $f : X \to S$ be a formal algebraic space of finite type. By $X_{\text{ét}}$, we denote the étale topos associated to the small étale site of X. Then we define the Zarski-Riemann space $ZR(X)$ of X by

$$ZR(X) = \varprojlim_{\pi : X' \to X, \pi \in B_X} (X')_{\text{ét}}.$$

Here the limit is the 2-projective limit in the category of local ringed topoi, and hence $ZR(X)$ is a local ringed topos. Since objects in B_X and morphisms are coherent, the projective limit is also coherent as a topos. Recall that, for a topos T, a point of T is a geometric morphism $p : \text{Sets} \to T$. By Deligne's theorem (the existence of sufficiently many points for a coherent topos [SGA4-2]), $ZR(X)$ has sufficiently many points, and we have the following proposition:

Proposition 3.1. *Let S be a noetherian formal scheme, $f : X \to S$ a formal algebraic space of finite type. Then $ZR(X)$ is a coherent topos, with sufficiently many points.*

More intuitively, points of $ZR(X)$ are described as follows as in the formal scheme case: By the definition, a point of $ZR(X)$ is described as a compatible system $\{x_{X'}\}_{X' \in BL_X}$, where $x_{X'}$ is a geometric point of $(X')_{\text{ét}}$ as a topos. Any point of $(X')_{\text{ét}}$ as a topos corresponds a geometric point of X' (i.e., a morphism $\operatorname{Spec} k \to X'$ where k is separably closed).

As in the formal scheme case, any morphism $\operatorname{Spf} V \to X$, where V is a strict henselian valuation ring which is complete for IV-adic topology, defines a point of $ZR(X)$.

It is verified that any point of $ZR(X)$ is obtained from some strict henselian valuation ring V and a morphism $\operatorname{Spf} V \to X$.

4. Equivalence theorem

Recall the equivalence theorem in the noetherian case:

Theorem 4.1. *(Equivalence theorem) Fix a noetherian formal scheme S, and a formal algebraic space X of finite type over S. Then there is an admissible blow up $X' \to X$ such that X' is a formal scheme.*

First we reduce the proof of theorem 4.1 to the following lemma by Nagata's termination argument using the coherence of $ZR(X)$:

Lemma 4.2. *(Key lemma) Assumptions are as in theorem 4.1. Take any point x of $ZR(X)$ corresponding to $\eta : \operatorname{Spf} V \to X$, where V is a strict henselian valuation ring which is complete for IV-adic topology. Then there is an open subspace U_x of some admissible blow up X' of X such that*

- *$\eta \to X_{\text{ét}}$ is factored into $\eta \to (U_x)_{\text{ét}} \to X_{\text{ét}}$.*
- *U_x is a formal scheme.*

Assume the Key lemma. For any point x of $ZR(X)$, we take an open subspace U_x of some admissible blow up of X given by the lemma. Then $\{(U_x)^{\text{rig}}\}_{x\text{: points of } ZR(X)}$ covers $ZR(X)$. By the quasi-compactness,

$$ZR(X) = \bigcup_{i=1}^{N} (U_{x_i})^{\text{rig}}.$$

for a finite number of points $x_1, \ldots, x_N$. By replacing U_{x_i} if necessary, we may assume that there is an admissible blow up X' with an invertible ideal of definition, and any U_{x_i} is an open subspace of X'. Since $(U_{x_i})^{\text{rig}}$ covers $ZR(X)$ and X' admits an invertible ideal of definition, $X' = \bigcup_{i=1}^{N} U_{x_i}$, and hence $(X')_0$ is a scheme since it is covered by open subspaces which are schemes. By Knudsen's theorem recalled in §3, X' is a formal scheme.

5. Open interior trick

We prove the Key lemma using Néron blow ups and Chow's lemma for algebraic spaces.

In this section, we assume that the height of V is one. Assume that I is invertible, and $\mathscr{O}_X$ has no I-torsions.

Take a stratification $Z_0 = X_0 \supset Z_1 \supset \cdots Z_N \supset Z_{N+1} = \emptyset$ where Z_α is finitely presented closed subspace, and each strata $W_\alpha = Z_\alpha \setminus Z_{\alpha+1}$ is a scheme over S_0.

$$X_0 = \coprod_{0 \le \alpha \le N} W_\alpha.$$

Assume rigid point $\eta : \operatorname{Spf} V \to X$ lies above W_β. We may remove $Z_{\beta+1}$ from X, and assume that Z_β is closed in X_0. Let J be the ideal defining Z_β. J is finitely generated. Let $Z \to X$ be the blow up of (I, J^n) for $n \ge 1$, W be the open part of Z where $I\mathscr{O}_Z$ generates $(I, J^n)\mathscr{O}_Z$.

First we prove that W_0 is a scheme. The projection $W_0 \xrightarrow{p} X_0$ is quasi-projective, and the fibers over $X_0 \setminus Z_\beta$ is empty. Let $H = \ker(\mathscr{O}_{X_0} \to p_* \mathscr{O}_{W_0})$ be the ideal defining the scheme theoretical image of p. Since $H|_{X_0 \setminus Z_\beta} = \mathscr{O}|_{X_0 \setminus Z_\beta}$, there is $m \ge 1$ such that $J^m \subset H$. Let C be the closed subspace of X_0 defined by J^m. Then $W_0 \xrightarrow{p} X_0$ is factorized as $W_0 \xrightarrow{p_0} C \overset{i}{\hookrightarrow} X_0$. Since i is a closed immersion and p is quasi-projective, p_0 is also quasi-projective. Since C is a nilpotent thickning of Z_β, C is a scheme by Knudson's theorem (cf. §3). It also follows that W_0 is a scheme, and hence W is a formal scheme.

It is clear that η lifts to W if n is sufficiently large. So the Key lemma is shown in the height one case.

Remark 5.1. *For two ideals I, J on a scheme X, $\pi : X' = Bl((I, J)) \to X$ be the blow up of X with respect to (I, J). Let Y be the open subspace of X where $I\mathscr{O}_Y$ generates $(I, J)\mathscr{O}_Y$. Y is called the Néron blow up of X. In the proof, we have used the notion of tube in rigid geometry, and the fact that these tubes are approximated by Néron blowing ups.*

6. Closure trick

We prove the general case of the Key lemma. Let η' be the height one point associated to $\eta : \operatorname{Spf} V \to X$ (the point defined by the associated height one valuation of V). By taking a blow up, we may assume that there is a quasi-compact open subspace U of X which is an affine formal scheme, and η' sits inside U_0.

Since the height one case is already shown, after replacing X by an admissible blow up if necessary, we may assume that the closure $\overline{U_0}$ in X_0 is a scheme.

Lemma 6.1. *There is an admissible blow up $\tilde{X} \to X$ centered in $X_0 \setminus U_0$ such that*

1. *There is an invertible admissible ideal J on $\tilde{X}$ which define a Cartier divisor on $\overline{U_0}$ whose support is $\overline{U_0} \setminus U_0$ set-theoretically.*
2. *$I\mathscr{O}_{\tilde{X}_0 \setminus \overline{U_0}}$ is zero.*

Proof. By Chow's lemma for algebraic spaces (cf. §3), there is an admissible blow up $X' \to X$ which is centered on $X_0 \setminus U_0$ such that the closure $\overline{U_0}$ in X' is a scheme. Since U_0 is quasi-compact, there is a finitely presented closed subscheme Z_0 of X_0 whose support is $\overline{U_0} \setminus U_0$ set-theoretically. By blowing up Z_0 and taking the strict transform of $\overline{U_0}$, we may assume that Z_0 is a Cartier divisor. By taking some mutiple of Z_0 and lifting the defining ideal I_{Z_0} in $\overline{U_0}$ to an admissible ideal I' of X', we get the desired $\tilde{X} \to X$ by blowing up I'. □

By applying lemma 6.1, we may assume that there is an admissible invertible ideal J on X which satisfies the conclusion of lemma 6.1. For $n \ge 2$, consider the blow up $Z \to X$ of (I, J^n), and V denotes the part where $J^n\mathscr{O}_Z$ generate $(I, J^n)\mathscr{O}_Z$. V contains U canonically, and the fiber of $V_0 \to X_0$ over $X_0 \setminus \overline{U_0}$ is empty. It is easy to see $V_0 \to X_0$ factors through $V_0 \to \overline{U_0} \hookrightarrow X_0$. Since V_0 is affine over X_0, $V_0 \to \overline{U_0}$ is also affine. It follows that V_0 is a scheme since $\overline{U_0}$ is, and hence V is a formal scheme. It is easy to see that η is inside V^{rig}, and the claim follows.

References

[1] Bosch, S.; Güntzer, U.; Remmert, R.: *Non-Archimedean analysis. A systematic approach to rigid analytic geometry*, Grundlehren der Mathematischen Wissenschaften, 261, Springer-Verlag, Berlin, 1984.

[2] Conrad, B.; Temkin, M.: Non-archimedian analytification of algebraic spaces, Journal of Algebraic geometry, to appear.

[3] Fujiwara, K.; Kato, F.: *Rigid geometry and applications*, in Moduli spaces and arithmetic geometry, Advanced Studies in Pure Math., vol 45 (2006), 327–386

[4] Fujiwara, K.; Kato, F.: *Foundations of rigid geometry*, in preparation.

[5] Knutson D.: *Algebraic Spaces*, Lecture Notes in Mathematics, Vol. 203, (1971), Springer-Verlag.

[6] Raynaud, M.: *Géométrie analytique rigide d'après Tate, Kiehl,*$\cdots$, Table Ronde d'Analyse non archimédienne (Paris, 1972), 319–327, Bull. Soc. Math. France, Mem. No. **39–40**, Soc. Math. France, Paris, 1974.

[7] Raynaud, M.; Gruson, L.: *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[8] Tate, J.: *Rigid analytic spaces*, Invent. Math. **12** (1971), 257–289.

[EGA] Grothendieck, A., Dieudonné, J.: *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math., no. **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32**, 1961-1967.

[EGA, $\mathrm{I}_{\mathrm{new}}$] Grothendieck, A., Dieudonné, J.: *Éléments de géométrie algébrique I.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 166, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[SGA4-2] *Théorie des topos et cohomologie étale des schémas. Tome* 2. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics, Vol. **270**. Springer-Verlag, Berlin-New York, 1972.