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MACKEY-FUNCTOR STRUCTURE ON THE BRAUER GROUPS
OF A FINITE GALOIS COVERING OF SCHEMES

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Abstract. For any finite étale covering of schemes, we can associate two homomorphisms for Brauer groups, namely the pull-back and the norm map. These homomorphisms make Brauer groups into a bivariant functor (a Mackey functor). Restricting to a finite Galois covering of schemes, we obtain a cohomological Mackey functor on its Galois group. This is a generalization of the result for rings by Ford [5]. Applying Bley and Boltje’s theorem [1], we can derive certain isomorphisms for the Brauer groups of intermediate coverings.

1. Introduction

In this paper, a scheme $S$ is always assumed to be Noetherian, and $\pi(S)$ denotes its étale fundamental group. Since we use Čech cohomology, we assume $S$ satisfies the following:

Assumption 1.1. For any finite set $E$ of points of $S$, there exists an open set $U \subset S$, such that $U$ contains every point in $E$.

As for the étale fundamental group and related notion, we follow the terminology in [9]. For example a finite étale covering is just a finite étale morphism of schemes.

Our aim is to make the following generalization of the result for rings by Ford [5].

Corollary (Corollary 4.2). Let $\pi : Y \to X$ be a finite Galois covering of schemes with Galois group $G$. Then the correspondence

$$H \leq G \mapsto \Br(Y/H)$$

forms a cohomological Mackey functor on $G$.

This follows from our main theorem;

Theorem (Theorem 3.5). Let $S$ be a connected Noetherian scheme. Let $(\Fet/S)$ denote the category of finite étale coverings over $S$. Then, the Brauer group functor $\Br$ forms a cohomological Mackey functor on $(\Fet/S)$.

As in Definition 3.1, a Mackey functor is a bivariant pair of functors $\Br = (\Br^*, \Br_*)$. For any morphism $\pi : Y \to X$, the contravariant part $\Br^*(\pi) : \Br(X) \to \Br(Y)$ is the pull-back, and $\Br_*(\pi) : \Br(Y) \to \Br(X)$ is the norm map defined later.

By applying Bley and Boltje’s theorem (Fact 5.2) to Corollary 4.2, we can obtain certain relations between Brauer groups of intermediate coverings:

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Corollary (Corollary 5.3). Let $X$ be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with $\text{Gal}(Y/X) = G$.

(i) Let $\ell$ be a prime number. If $H \leq G$ is not $\ell$-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules\[
\bigoplus_{U = H_0 < \ldots < H_n = H} \text{Br}(Y/U)(\ell)[U] \cong \bigoplus_{U = H_0 < \ldots < H_n = H} \text{Br}(Y/U)(\ell)[U],
\]

(ii) If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups\[
\bigoplus_{U = H_0 < \ldots < H_n = H} \text{Br}(Y/U)[U] \cong \bigoplus_{U = H_0 < \ldots < H_n = H} \text{Br}(Y/U)[U].
\]

2. Restriction and corestriction

Remark 2.1. For any scheme $X$, there exists a natural monomorphism \[\chi_X : \text{Br}(X) \hookrightarrow \text{Br}'(X) := H^2_{\text{et}}(X, \mathbb{G}_m,X)_{\text{tor}},\]
such that for any morphism $\pi : Y \to X$,

\[
\begin{array}{ccc}
\text{Br}(X) & \xrightarrow{\pi^*} & \text{Br}(Y) \\
\downarrow{\chi_X} & & \downarrow{\chi_Y} \\
H^2_{\text{et}}(X, \mathbb{G}_m,X) & \xrightarrow{\pi^*} & H^2_{\text{et}}(Y, \mathbb{G}_m,Y)
\end{array}
\]
is a commutative diagram.

Here $\pi^* : \text{Br}(X) \to \text{Br}(Y)$ is the pull-back of Azumaya algebras, while $\pi^* : H^2_{\text{et}}(X, \mathbb{G}_m,X) \to H^2_{\text{et}}(Y, \mathbb{G}_m,Y)$ is defined as the composition of the canonical morphism

\[H^2_{\text{et}}(X, \mathbb{G}_m,X) \to H^2_{\text{et}}(Y, \pi_* \mathbb{G}_m,Y)\]
and

\[H^2_{\text{et}}(\pi_* \mathbb{G}_m,X) \to H^2_{\text{et}}(X, \pi_* \mathbb{G}_m,Y),\]

where $\pi_* : \mathbb{G}_m,X \to \pi_* \mathbb{G}_m,Y$ is the canonical (structural) homomorphism of étale sheaves on $X$. We call these $\pi^*$ the restriction maps.

Remark 2.2. For any finite étale covering $\pi : Y \to X$, there exists a homomorphism of étale sheaves on $X$

\[N_{Y/X} : \pi_* \mathbb{G}_m,Y \to \mathbb{G}_m,X\]
which induces the norm map for finite étale ring extensions.

When $\pi : Y \to X$ is a finite étale covering, the canonical homomorphism $H^2_{\text{et}}(X, \pi_* \mathbb{G}_m,Y) \to H^2_{\text{et}}(Y, \mathbb{G}_m,Y)$ becomes isomorphic (cf. [6]). By composing $H^2_{\text{et}}(N_{Y/X})$ with the inverse of this canonical isomorphism, we define the corestriction map for cohomology groups:

\[\text{cor}_\pi : H^2_{\text{et}}(Y, \mathbb{G}_m,Y) \xrightarrow{\cong} H^2_{\text{et}}(X, \pi_* \mathbb{G}_m,Y) \xrightarrow{H^2_{\text{et}}(N_{Y/X})^{-1}} H^2_{\text{et}}(X, \mathbb{G}_m,X).\]
Proposition 2.3. Let \( \pi : Y \to X \) as before. There exists a corestriction homomorphism for Brauer groups

\[
\text{cor}_\pi : \text{Br}(Y) \to \text{Br}(X),
\]

such that

\[
\begin{array}{ccc}
\text{Br}(Y) & \xrightarrow{\text{cor}_\pi} & \text{Br}(X) \\
\downarrow \chi_Y & & \downarrow \chi_X \\
H^2_{\text{et}}(Y, \mathbb{G}_m,Y) & \xrightarrow{\text{cor}_\pi} & H^2_{\text{et}}(X, \mathbb{G}_m,X)
\end{array}
\]

is commutative.

To construct \( \text{cor} : \text{Br}(Y) \to \text{Br}(X) \), we define a monoidal functor

\[
N_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X).
\]

Lemma 2.4. Let \( \pi : Y \to X \) be a finite étale covering of constant degree \( d \). There exists a monoidal functor (unique up to a natural isomorphism)

\[
N_{\pi} = N_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X),
\]

\((\text{q-Coh}(X) : \text{the category of quasi-coherent Zariski sheaves on } X)\), such that

(i) When \( Y \) is isomorphic to a disjoint union of \( d \)-copies of \( X \), i.e. when \( Y = \coprod_{1 \leq i \leq d} Y_i \) and \( \exists \eta_i : X \xrightarrow{\sim} Y_i \), then \( N_{Y/X} \) is defined by

\[
N_{Y/X}(F) := \eta_1^*(F|_{Y_1}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \eta_d^*(F|_{Y_d}) \quad (\forall F \in \text{q-Coh}(Y)).
\]

(ii) For any pull-back by a morphism \( f : X' \to X \)

\[
\begin{array}{ccc}
Y' & \xrightarrow{\pi'} & X' \\
g \downarrow & \square & f \\
Y & \xrightarrow{\pi} & X
\end{array}
\]

there exists a natural isomorphism of monoidal functors

\[
N_{Y'/X'} \circ g^* \cong f^* \circ N_{Y/X}.
\]

Proof. When \( Y \) is isomorphic to a disjoint union of \( d \)-copies of \( X \), then \( N_{Y/X} \) is defined by as in (i).

For a general case, remark that

\[
Y' \cong \coprod_{1 \leq i \leq d} X' \quad (\forall F \in \text{q-Coh}(Y))
\]

\[
\begin{array}{ccc}
Y' & \xrightarrow{\pi'} & X' \\
g \downarrow & \square & f \\
Y & \xrightarrow{\pi} & X
\end{array}
\]

For any \( F \in \text{q-Coh}(Y) \), put \( \mathcal{F} := N_{Y'/X'}(g^*(F)) \). Then \( \mathcal{F} \) descends to yield \( N_{Y/X}(F) \in \text{q-Coh}(X) \). Thus we obtain a monoidal functor \( N_{Y/X} \). This construction does not depend on the choice of \( f \), up to an isomorphism of monoidal functors. By the reduction to the disjoint-union case as above, we can show (ii). \( \square \)
While this \( N_{Y/X} \) is a generalization of the norm functor for a finite étale ring extension (Knus-Ojanguren [8], Ferrand [4]), it is also possible to define \( N_{Y/X} \) by gluing those for affines.

**Lemma 2.6.** Let \( \pi : Y \to X \) be a finite étale covering of constant degree \( d \). \( N_{Y/X} \) has the following properties:

1. \( N_{Y/X} \) is monoidal.
2. For any \( F, G \in \text{q-Coh}(Y) \), there exists a functorial morphism
   \[
   \theta_{Y/X} : N_{Y/X}(\text{Hom}_{O_Y}(F, G)) \to \text{Hom}_{O_X}(N_{Y/X}(F), N_{Y/X}(G)).
   \]

(1\(^+\)) Moreover if \( G \) is locally free of finite rank, this is an isomorphism.

3. There exists a natural isomorphism
   \[
   N_{Y/X}(O_Y^\otimes n) \cong O_X^\otimes nd.
   \]

(2\(^+\)) More generally, if \( F \) is locally free \( O_Y \)-module of finite rank \( n \), then \( N_{Y/X}(F) \) becomes locally free \( O_X \)-module of rank \( nd \).

For a general (non-constant degree) \( \pi : Y \to X \), we can define the norm functor on each connected component of \( X \) as above, and glue them to obtain the norm functor \( N_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X) \).

**Proof.** Conditions (0) and (2) follow from the definition of \( N_{Y/X} \). By taking an affine cover \( X = \bigcup_{i \in I} U_i \), (2\(^+\)) reduces to the case where \( X, Y \) are affine, shown by Ferrand [4]. As for condition (1), existence of \( \theta_{Y/X} \) simply follows from the fact that \( N_{Y/X} \) is a monoidal functor between closed symmetric monoidal categories. (1\(^+\)) is shown by a reduction to the affine case. \( \square \)

**Proof.** (proof of Proposition) By the above lemma, especially we have an isomorphism

\[
N_{Y/X}(M_n(O_Y)) \cong M_{nd}(O_X)
\]

of \( O_X \)-algebras, for any finite étale covering \( Y/X \) of constant degree \( d \).

Remark that for any \( O_Y \)-algebra \( A \) of finite type, \( A \) is an Azumaya algebra if and only if \( A \) is étale locally isomorphic to \( M_n(O_Y) \). Thus for any Azumaya algebra \( A \), there exists a covering \( V := \{ V_i \xrightarrow{g_i} Y \}_{i \in I} \) of \( Y \) in the étale topology (simply written ‘\( V \in \text{Cov}_{\text{ét}}(Y) \)’) such that

\[
g_i^* A \cong M_{n_i}(O_{V_i}) \quad (\exists n_i \in \mathbb{N}).
\]

Replacing \( V \) by its refinement, we may assume that there exists a covering \( U := \{ U_i \xrightarrow{f_i} X \}_{i \in I} \in \text{Cov}_{\text{ét}}(X) \) such that

\[
V = \pi^* U := \{ Y \times_X U_i \xrightarrow{pr_Y} Y \}_{i \in I}.
\]

Then we have \( f_i^* N_{Y/X}(A) \cong \tilde{N}_{V_i/U_i}(g_i^* A) \cong M_{n_i}(O_{U_i}) \). Thus \( N_{Y/X}(A) \) also becomes an Azumaya algebra.

By the isomorphism

\[
N_{Y/X}(\text{End}(E)) \cong \text{End}(N_{Y/X}(E)) \quad (\forall E : \text{locally free of finite rank})
\]

and the monoidality of \( N_{Y/X} \), we obtain a well-defined homomorphism

\[
\text{cor}_\pi : \text{Br}(Y) \xrightarrow{\psi} \text{Br}(X) \quad \psi
\]

\[
A \xrightarrow{\varphi} N_{Y/X}(A).
\]
By using Čech cohomology, we can show the commutativity of
\[ \begin{align*}
\Br(Y) @>\text{cor}>> \Br(X) \\
H^2(\alpha Y, \mathbb{G}_m, Y) @>\text{cor}>> H^2(\alpha X, \mathbb{G}_m, X).
\end{align*} \]

3. BRAUER-GROUP MACKEY FUNCTOR

For any profinite group \(G\), let (fin. \(G\)-space) denote the category of finite discrete \(G\)-spaces and equivariant \(G\)-maps.

**Definition 3.1.** Let \(C\) be a Galois category, with fundamental functor \(F\) (i.e. there exists a profinite group \(\pi(C)\) such that \(F\) gives an equivalence from \(C\) to (fin. \(\pi(C)\)-space)).

A cohomological Mackey functor on \(C\) is a pair of functors \(M=\left( M^*, M_* \right) \) from \(C\) to \(\text{Ab}\), where \(M^*\) is contravariant and \(M_*\) is covariant, satisfying the following conditions:

1. (Additivity) For each coproduct \(X_i \rightarrow X \amalg Y_i \leftarrow Y\) in \(C\), the canonical morphism
   \[ (M^*(i_X), M^*(i_Y)) : M(X \amalg Y) \rightarrow M(X) \oplus M(Y) \]
   is an isomorphism.
2. (Mackey condition) For any pull-back diagram
   \[ \begin{array}{ccc} 
   Y' & \xrightarrow{\varpi'} & Y \\
   \downarrow{\pi} & \nearrow{\varpi} & \downarrow{\pi'} \\
   X' & \xrightarrow{\varpi} & X, 
   \end{array} \]
   the following diagram is commutative:
   \[ \begin{array}{ccc}
   M(Y) & \xrightarrow{M^*(\varpi')} & M(Y') \\
   \downarrow{M_*(\pi)} & \nearrow{M_*(\varpi')} & \downarrow{M_*(\pi')} \\
   M(X) & \xrightarrow{M^*(\varpi)} & M(X'). 
   \end{array} \]
3. (Cohomological condition) For any morphism \(\pi : X \rightarrow Y\) in \(C\) with \(X\) and \(Y\) connected, we have
   \[ M_*(\pi) \circ M^*(\pi) = \text{multiplication by } \deg(\pi) \]
   where \(\deg(\pi) := \#F(Y)/\#F(X)\).

\[ \begin{array}{ccc}
M(X) & \xrightarrow{M^*(\varpi)} & M(Y) \\
\downarrow{\deg \pi} & \nearrow{M_*(\varpi)} & \downarrow{\deg \pi} \\
M(X) & \xrightarrow{} & M(X). 
\end{array} \]
A standard example is the cohomological Mackey functor on a profinite group $G$ (in terminology of [1], a cohomological Mackey functor on the finite natural Mackey system on $G$):

**Definition 3.2.** Let $G$ be a profinite group, and put $\mathcal{C} := (\text{fin. } G\text{-space})$, $F := \text{id.}$ A cohomological Mackey functor on $\mathcal{C}$ is simply called a cohomological Mackey functor on $G$, and their category is denoted by $\mathcal{Mack}_c(G)$.

**Remark 3.3.** Since any object $X$ in (fin. $G$-space) is a direct sum of transitive $G$-sets of the form $G/H$ where $H$ is a open subgroup of $G$, a Mackey functor on $G$ is equivalent to the following datum:

- an abelian group $M(H)$ for each open $H \leq G$, with structure maps
- a homomorphism $\text{res}_{H}^{K} : M(H) \to M(K)$ for each open $K \leq H \leq G$,
- a homomorphism $\text{cor}_{H}^{K} : M(K) \to M(H)$ for each open $K \leq H \leq G$,
- a homomorphism $\text{c}_{g,H} : M(H) \to M(gH)$ for each open $H \leq G$ and $g \in G$,

where $gH := ghg^{-1}$, satisfying certain compatibilities (cf. [1]). Here $M(G/H)$ is abbreviated to $M(H)$ for any open $H \leq G$.

**Example 3.4.** In this notation, for any $G$-module $A$ and any $n \geq 0$, the group cohomology

$$H \mapsto H^n(H, A) \quad (\forall H \leq G \text{ open})$$

becomes a cohomological Mackey functor on $G$, with appropriate structure maps.

For any finite étale covering $\pi : Y \to X$, put $\text{Br}^*(\pi) := \text{res}_{\pi}$ and $\text{Br}_*(\pi) := \text{cor}_{\pi}$. Then we obtain a cohomological Mackey functor $\text{Br}$ (and similarly $\text{Br}'$, $\text{H}^2_{et}(-, \mathbb{G}_m)$):

**Theorem 3.5.** For any connected Noetherian scheme $S$, we have a sequence of cohomological Mackey functors on (F\text{Et}/$S$)

$$\text{Br} \leftrightarrow \text{Br}' \leftrightarrow \text{H}^2_{et}(-, \mathbb{G}_m).$$

**Proof.** We only show Mackey and cohomological conditions. Since restrictions and corestrictions are compatible with inclusions

$$\text{Br}(X) \leftrightarrow \text{Br}'(X) \leftrightarrow \text{H}^2_{et}(X, \mathbb{G}_m),$$

it suffices to show for $\text{H}^2_{et}(-, \mathbb{G}_m)$.

(Mackey condition) For any pull-back diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\omega'} & Y' \\
\pi & \downarrow & \pi' \\
X & \xleftarrow{\omega} & X'
\end{array}$$

in (F\text{Et}/$S$), we have a commutative diagram

$$\begin{array}{ccc}
\pi_*\mathbb{G}_{m,Y} & \xrightarrow{\pi_*(\omega'_1)} & \pi_*\omega'_*\mathbb{G}_{m,Y'} \\
N_{Y'/X} \downarrow & & \downarrow \omega_* \pi'_*\mathbb{G}_{m,Y'} \\
\mathbb{G}_{m,X} & \xrightarrow{\omega_1} & \omega_*\mathbb{G}_{m,X'}
\end{array}$$

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This yields a commutative diagram

\[
\begin{array}{c}
H^2_{\text{et}}(X, G_{m,X}) \\
\downarrow \text{cor}\pi \\
H^2_{\text{et}}(Y, G_{m,Y})
\end{array}
\rightarrow
\begin{array}{c}
\circlearrowright \\
H^2_{\text{et}}(Y', G_{m,Y'})
\end{array}
\rightarrow
\begin{array}{c}
H^2_{\text{et}}(X', G_{m,X'}) \\
\downarrow \text{cor}\pi'
\end{array}
\] \hspace{1cm}
\begin{array}{c}
H^2_{\text{et}}(X, G_{m,X}) \\
\downarrow \text{res}_{\pi}
\end{array}
\rightarrow
\begin{array}{c}
\circlearrowright \\
H^2_{\text{et}}(X', G_{m,X'})
\end{array}
\]

(Cohomological condition) For any morphism \( \pi : Y \to X \) in (\( \text{FEt}/S \)) with \( X \) and \( Y \) connected, since

\[ N_{Y/X} \circ \pi : G_{m,Y} \to G_{m,X} \]

is equal to the multiplication by \( d = \deg(\pi) \)

\[ \begin{array}{ccc}
\pi_+ & \pi_+ G_{m,Y} & N_{Y/X} \\
& \circlearrowright & d \\
G_{m,X} & \to & G_{m,X},
\end{array} \]

we obtain \( \text{cor}_{\pi} \circ \text{res}_{\pi} = d \).

4. Restriction to a finite Galois covering

Thus we have obtained a Mackey functor \( \text{Br} \) on (\( \text{FEt}/S \)). By pulling back by a quasi-inverse \( S \) of the fundamental functor

\[ F : (\text{FEt}/S) \to (\text{fin. } \pi(S)-\text{space}), \]

we can obtain a Mackey functor on \( \pi(S) \):

**Corollary 4.1.** There is a sequence of cohomological Mackey functors

\[ \text{Br} \circ S \leftarrow \text{Br}' \circ S \leftarrow H^2_{\text{et}}(-, G_{m}) \circ S \]

on \( \pi(S) \), where \( \text{Br} \circ S := (\text{Br}^* \circ S, \text{Br}_* \circ S) \) and so on.

**Corollary 4.2.** Let \( X \) be a connected Noetherian scheme. For any finite Galois covering \( \pi : Y \to X \) with \( \text{Gal}(Y/X) = G \), there exists a cohomological Mackey functor \( \text{Br} \) on \( G \) which satisfies

\[ \text{Br}(H) \cong \text{Br}(Y/H) \quad (\forall H \leq G), \]

with structure maps induced from restrictions and corestrictions of Brauer groups. (We abbreviate \( \text{Br}(G/H) \) to \( \text{Br}(H) \).)
Proof. By the previous corollary, we have a cohomological Mackey functor $Br \circ S$ on $\pi(X)$. Since there is a projection $pr : \pi(X) \to G^{op}$, we can regard any finite $G^{op}$-set naturally as a finite $\pi(X)$-space, to obtain a functor 

$$(\text{fin. } G^{op}\text{-set}) \to (\text{fin. } \pi(X)\text{-space}).$$

Pulling back by this functor, and taking the opposite Mackey functor, we obtain

$$M \Rightarrow \text{Mack}(\pi(X)), \text{Mack}(G^{op}), \text{Mack}(G) \in \text{Mack}(G).$$

In terms of subgroups of $G$, $M_G$ satisfies

$$M_G(H) = M(pr^{-1}(H^{op})) \quad (\forall H \leq G).$$

Applying this to $Br \circ S$, we obtain $Br := (Br \circ S)_G \subseteq \text{Mack}(G)$. Since the equivalence $S : (\text{fin. } \pi(X)\text{-space}) \xrightarrow{\sim} (\text{FEt}/X)$ satisfies

$$S(\pi(X)/pr^{-1}(H^{op})) \cong Y/H,$$

we have

$$Br(Y/H) \cong Br(Y/H).$$

Similarly we can define $Br'$ (and also $(H_2^{et}(\cdot, \mathbb{G}_m) \circ S)_G$). Since $\text{Mack}(G)$ is an abelian category with objectwise (co-)kernels (see for example [3]), we can take the quotient Mackey functor $Br'/Br \in \text{Mack}(G)$, which satisfies

$$(Br'/Br)(H) \cong (Br'(Y/H))/(Br(Y/H)).$$

5. Application of Bley and Boltje’s theorem

Let $\ell$ be a prime number. For any abelian group $A$, let

$$A(\ell) := \{m \in A \mid \exists e \in \mathbb{N}_{\geq 0}, \ell^em = 0\}$$

be the $\ell$-primary part. This is a $\mathbb{Z}_\ell$-module.

Definition 5.1 ([1]). For any finite group $H$, $H$ is $\ell$-hypoelementary $\iff H$ has a normal $\ell$-subgroup with a cyclic quotient.

$H$ is hypoelementary $\iff H$ is $\ell$-hypoelementary for some prime $\ell$.

Fact 5.2 ([1]). Let $M$ be a cohomological Mackey functor on a finite group $G$.

(i) Let $\ell$ be a prime number. If $H \leq G$ is not $\ell$-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n \text{ odd}}} M(U)(\ell)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n \text{ even}}} M(U)(\ell)^{|U|}.$$

(ii) If $H \leq G$ is not hypoelementary and $M(U)$ is torsion for any subgroup $U \leq H$, then there is a natural isomorphism of abelian groups

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n \text{ odd}}} M(U)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n \text{ even}}} M(U)^{|U|}.$$

Here, $|U|$ denotes the order of $U$. 

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Applying this theorem to \( \text{Br} \), we obtain the following relations for the Brauer groups of intermediate étale coverings:

**Corollary 5.3.** Let \( X \) be a connected Noetherian scheme and \( \pi : Y \to X \) be a finite Galois covering with \( \text{Gal}(Y/X) = G \).

(i) Let \( \ell \) be a prime number. If \( H \leq G \) is not \( \ell \)-hypoelementary, then there is a natural isomorphism of \( \mathbb{Z}_\ell \)-modules

\[
\bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)(\ell)^{[U]} \cong \bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)(\ell)^{[U]}.
\]

(ii) If \( H \leq G \) is not hypoelementary, then there is a natural isomorphism of abelian groups

\[
\bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)^{[U]} \cong \bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)^{[U]}.
\]

Finally, we derive some numerical equations related to Brauer groups from Corollary 5.3.

**Definition 5.4.** Let \( G \) be a finite group. For any subgroups \( U \leq H \leq G \), put

\[
\mu(U, H) := \sum_{U = H_0 < \cdots < H_n = H} (-1)^n, \quad \text{Möbius function}.
\]

If \( m \) (resp. \( m_\ell \)) is an additive invariant of abelian groups (resp. \( \mathbb{Z}_\ell \)-modules) which is finite on Brauer groups, we obtain the following equations:

**Corollary 5.5.** Let \( \pi : Y \to X \) as before, \( G = \text{Gal}(Y/X) \).

(i) If \( H \leq G \) is not \( \ell \)-hypoelementary,

\[
\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m_\ell(\text{Br}(Y/U)(\ell)) = 0.
\]

(ii) If \( H \leq G \) is not hypoelementary,

\[
\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m(\text{Br}(Y/U)) = 0.
\]

For a prime \( \ell \) and an abelian group \( A \), its corank is defined as \( \text{rank}_{\mathbb{Z}_\ell}(T_\ell(A)) \), where \( T_\ell(A) = \lim_n \text{Ker}(\ell^n : A \to A) \). In this note, we denote this by

\[
\text{rk}_\ell(A) := \text{rank}_{\mathbb{Z}_\ell}(T_\ell(A)).
\]

\( \text{Br}(X)(\ell) \) is known to be of finite corank, for example in the following cases ([7]):

- (C1) \( k \): a separably closed or finite field, \( X \): of finite type /\( k \), and proper or smooth /\( k \), or char(\( k \)) = 0 or dim \( X \) \leq 2.
- (C2) \( X \): of finite type /Spec(\( \mathbb{Z} \)), and smooth /Spec(\( \mathbb{Z} \)) or proper over \( \mathbb{Z} \)open \( \subset \text{Spec}(\mathbb{Z}) \).

Remark that if \( Y/X \) is a finite étale covering and \( X \) satisfies (C1) or (C2), then so does \( Y \).

**Example 5.6.** Assume \( X \) satisfies (C1) or (C2). For any non-\( \ell \)-hypoelementary subgroup \( H \leq G \), we have an equation

\[
\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot \text{rk}_\ell(\text{Br}(Y/H)(\ell)) = 0.
\]
Another example is related with the comparison of $\text{Br}$ and $\text{Br}'$. By Gabber’s lemma, for any finite étale covering $Y/X$, we have

$$\text{Br}'(X)/\text{Br}(X) \hookrightarrow \text{Br}'(Y)/\text{Br}(Y).$$

In particular, if $\text{Br}(Y) \subset \text{Br}(Y)'$ is of finite index, then so is $\text{Br}(X) \subset \text{Br}(X)'$.

**Example 5.7.** Assume $X$ satisfies $[\text{Br}'(Y) : \text{Br}(Y)] < \infty$. Then for any non-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U| \mu(U, H) \cdot [\text{Br}'(Y/U) : \text{Br}(Y/U)] = 0.$$

**References**


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