代数幾何学シンポジウム記録
2007年度 pp.91-100

MACKEY-FUNCTOR STRUCTURE ON THE BRAUER GROUPS OF A FINITE GALOIS COVERING OF SCHEMES

HIROYUKI NAKAOKA

Abstract. For any finite étale covering of schemes, we can associate two homomorphisms for Brauer groups, namely the pull-back and the norm map. These homomorphisms make Brauer groups into a bivariant functor (a Mackey functor). Restricting to a finite Galois covering of schemes, we obtain a cohomological Mackey functor on its Galois group. This is a generalization of the result for rings by Ford [5]. Applying Bley and Boltje's theorem [1], we can derive certain isomorphisms for the Brauer groups of intermediate coverings.

1. Introduction

In this paper, a scheme S is always assumed to be Noetherian, and $\pi(S)$ denotes its étale fundamental group. Since we use Čech cohomology, we assume S satisfies the following:

Assumption 1.1. For any finite set E of poits of S, there exists an open set $U \subset S$, such that U contains every point in E.

As for the étale fundamental group and related notion, we follow the terminology in [9]. For example a finite étale covering is just a finite étale morphism of schemes.

Our aim is to make the following generalization of the result for rings by Ford [5].

Corollary (Corollary 4.2)**.** Let $\pi : Y \to X$ be a finite Galois covering of schemes with Galois group G. Then the correspondence

$$H \leq G \mapsto \mathrm{Br}(Y/H)$$

forms a cohomological Mackey functor on G.

This follows from our main theorem;

Theorem (Theorem 3.5)**.** Let S be a connected Noetherian scheme. Let (FEt/S) denote the category of finite étale coverings over S. Then, the Brauer group functor Br forms a cohomological Mackey functor on (FEt/S).

As in Definition 3.1, a Mackey functor is a bivariant pair of functors $\mathrm{Br} = (\mathrm{Br}^*, \mathrm{Br}_*)$. For any morphism $\pi : Y \to X$, the contravariant part $\mathrm{Br}^*(\pi) : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is the pull-back, and $\mathrm{Br}_*(\pi) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is the norm map defined later.

By applying Bley and Boltje's theorem (Fact 5.2) to Corollary 4.2, we can obtain certain relations between Brauer groups of intermediate coverings:

The author wishes to thank Professor Kazuhiro Fujiwara for his useful comments.

Corollary (Corollary 5.3)**.** Let X be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with $\mathrm{Gal}(Y/X) = G$.
(i) Let ℓ be a prime number. If $H \leq G$ is not ℓ-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{odd}}} \mathrm{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{even}}} \mathrm{Br}(Y/U)(\ell)^{|U|}.$$

(ii) If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{odd}}} \mathrm{Br}(Y/U)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{even}}} \mathrm{Br}(Y/U)^{|U|}.$$

2. Restriction and corestriction

Remark 2.1. For any scheme X, there exists a natural monomorphism

$$\chi_X : \mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) := H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X})_{\mathrm{tor}} \ ,$$

such that for any morphism $\pi : Y \to X$,

$$\begin{tikzcd} \mathrm{Br}(X) \arrow[r, "\pi^*"] \arrow[d, hook, "\chi_X"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Br}(Y) \arrow[d, hook, "\chi_Y"] \\ H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \arrow[r, "\pi^*"] & H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \end{tikzcd}$$

is a commutative diagram.

Here $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is the pull-back of Azumaya algebras, while $\pi^* : H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \to H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y})$ is defined as the composition of the canonical morphism

$$H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}) \to H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y})$$

and

$$H^2_{\mathrm{et}}(\pi_\sharp) : H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \to H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}),$$

where $\pi_\sharp : \mathbb{G}_{m,X} \to \pi_*\mathbb{G}_{m,Y}$ is the canonical (structural) homomorphism of étale sheaves on X. We call these π^* the *restriction maps*.

Remark 2.2. For any finite étale covering $\pi : Y \to X$, there exists a homomorphism of étale sheaves on X

$$N_{Y/X} : \pi_*\mathbb{G}_{m,Y} \to \mathbb{G}_{m,X}$$

which induces the norm map for finite étale ring extensions.

When $\pi : Y \to X$ is a finite étale covering, the canonical homomorphism $H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}) \to H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y})$ becomes isomorphic (cf. [6]). By composing $H^2_{\mathrm{et}}(N_{Y/X})$ with the inverse of this canonical isomorphism, we define the *corestriction map* for cohomology groups:

$$\mathrm{cor}_\pi : H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \overset{\cong}{\to} H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}) \overset{H^2_{\mathrm{et}}(N_{Y/X})}{\longrightarrow} H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}).$$

Proposition 2.3. *Let $\pi : Y \to X$ as before. There exists a corestriction homomorphism for Brauer groups*

$$\mathrm{cor}_\pi : \mathrm{Br}(Y) \to \mathrm{Br}(X),$$

such that

$$\begin{tikzcd}
\mathrm{Br}(Y) \arrow[r, "\mathrm{cor}_\pi"] \arrow[d, hook, "\chi_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Br}(X) \arrow[d, hook, "\chi_X"] \\
H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \arrow[r, "\mathrm{cor}_\pi"'] & H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X})
\end{tikzcd}$$

is commutative.

To construct $\mathrm{cor} : \mathrm{Br}(Y) \to \mathrm{Br}(X)$, we define a monoidal functor

$$\mathcal{N}_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X).$$

Lemma 2.4. *Let $\pi : Y \to X$ be a finite étale covering of constant degree d. There exists a monoidal functor* (*unique up to a natural isomorphism*)

$$\mathcal{N}_\pi = \mathcal{N}_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X),$$

(q-Coh(X) : *the category of quasi-coherent Zariski sheaves on X*). *such that*
(i) *When Y is isomorphic to a disjoint union of d-copies of X, i.e. when $Y = \coprod_{1\le i\le d} Y_i$ and $\exists \eta_i : X \overset{\cong}{\to} Y_i$, then $\mathcal{N}_{Y/X}$ is defined by*

$$\mathcal{N}_{Y/X}(\mathcal{F}) := \eta_1^*(\mathcal{F}|_{Y_1}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \eta_d^*(\mathcal{F}|_{Y_d}) \quad (\forall \mathcal{F} \in \text{q-Coh}(Y)).$$

(ii) *For any pull-back by a morphism $f : X' \to X$*

$$\begin{tikzcd}
Y' \arrow[r, "\pi'"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "f"] \\
Y \arrow[r, "\pi"'] & X
\end{tikzcd}$$

there exists a natural isomorphism of monoidal functors

$$\mathcal{N}_{Y'/X'} \circ g^* \xrightarrow{\cong} f^* \circ \mathcal{N}_{Y/X}.$$

Proof. When Y is isomorphic to a disjoint union of d-copies of X, then $\mathcal{N}_{Y/X}$ is defined by as in (i).

For a general case, remark that

Remark 2.5. For any finite étale covering $\pi : Y \to X$ of constant degree d, there exists a fpqc morphism $f : X' \to X$ such that $Y \times_X X'$ is isomorphic to a disjoint union of d-copies of X'.

$$\begin{tikzcd}
Y' \cong \coprod_d X' \arrow[r, "\pi'"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "f"] \\
Y \arrow[r, "\pi"'] & X
\end{tikzcd}$$

For any $\mathcal{F} \in$ q-Coh(Y), put $\overline{\mathcal{F}} := \mathcal{N}_{Y'/X'}(g^*(\mathcal{F}))$. Then $\overline{\mathcal{F}}$ descends to yield $\mathcal{N}_{Y/X}(\mathcal{F}) \in$ q-Coh(X). Thus we obtain a monoidal functor $\mathcal{N}_{Y/X}$. This construction does not depend on the choice of f, up to an isomorphism of monoidal functors. By the reduction to the disjoint-union case as above, we can show (ii). □

While this $\mathcal{N}_{Y/X}$ is a generalization of the norm functor for a finite étale ring extension (Knus-Ojanguren [8], Ferrand [4]), it is also possible to define $\mathcal{N}_{Y/X}$ by gluing those for affines.

Lemma 2.6. *Let $\pi : Y \to X$ be a finite étale covering of constant degree d. $\mathcal{N}_{Y/X}$ has the following properties*:
(0) *$\mathcal{N}_{Y/X}$ is monoidal.*
(1) *For any $\mathcal{F}, \mathcal{G} \in \text{q-Coh}(Y)$, there exists a functorial morphism*

$$\theta_{Y/X} : \mathcal{N}_{Y/X}(\mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{F},\mathcal{G})) \to \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{N}_{Y/X}(\mathcal{F}),\mathcal{N}_{Y/X}(\mathcal{G})).$$

(1^+) *Moreover if $\mathcal{G}$ is locally free of finite rank, this is an isomorphism.*
(2) *There exists a natural isomorphism*

$$\mathcal{N}_{Y/X}(\mathcal{O}_Y^{\oplus n}) \cong \mathcal{O}_X^{\oplus n^d}.$$

(2^+) *More generally, if $\mathcal{F}$ is locally free $\mathcal{O}_Y$-module of finite rank n, then $\mathcal{N}_{Y/X}(\mathcal{F})$ becomes locally free $\mathcal{O}_X$-module of rank n^d.*

For a general (non-constant degree) $\pi : Y \to X$, we can define the norm functor on each connected component of X as above, and glue them to obtain the norm functor $\mathcal{N}_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X)$.

Proof. Conditions (0) and (2) follow from the definition of $\mathcal{N}_{Y/X}$. By taking an affine cover $X = \bigcup\limits_{i\in I} U_i$, ($2^+$) reduces to the case where X, Y are affine, shown by Ferrand [4]. As for condition (1), existence of $\theta_{Y/X}$ simply follows from the fact that $\mathcal{N}_{Y/X}$ is a monoidal functor between closed symmetric monoidal categories. (1^+) is shown by a reduction to the affine case. □

Proof. (proof of Proposition) By the above lemma, especially we have an isomorphism

$$\mathcal{N}_{Y/X}(\mathcal{M}_n(\mathcal{O}_Y)) \cong \mathcal{M}_{n^d}(\mathcal{O}_X)$$

of $\mathcal{O}_X$-algebras, for any finite étale covering Y/X of constant degree d.

Remark that for any $\mathcal{O}_Y$-algebra $\mathcal{A}$ of finite type, $\mathcal{A}$ is an Azumaya algebra if and only if $\mathcal{A}$ is étale locally isomorphic to $\mathcal{M}_n(\mathcal{O}_Y)$. Thus for any Azumaya algebra $\mathcal{A}$, there exists a covering $\mathcal{V} := \{V_i \xrightarrow{g_i} Y\}_{i\in I}$ of Y in the étale topology (simply written '$\mathcal{V} \in \text{Cov}_{\text{et}}(Y)$') such that

$$g_i^*\mathcal{A} \cong \mathcal{M}_{n_i}(\mathcal{O}_{V_i}) \quad (\exists n_i \in \mathbb{N}).$$

Replacing $\mathcal{V}$ by its refinement, we may assume that there exists a covering $\mathcal{U} = \{U_i \xrightarrow{f_i} X\}_{i\in I} \in \text{Cov}_{\text{et}}(X)$ such that

$$\mathcal{V} = \pi^*\mathcal{U} := \{Y \times_X U_i \xrightarrow{\text{pr}_Y} Y\}_{i\in I}.$$

Then we have $f_i^*\mathcal{N}_{Y/X}(\mathcal{A}) \cong \mathcal{N}_{V_i/U_i}(g_i^*\mathcal{A}) \cong \mathcal{M}_{n_i^d}(\mathcal{O}_{U_i})$. Thus $\mathcal{N}_{Y/X}(\mathcal{A})$ also becomes an Azumaya algebra.

By the isomorphism

$$\mathcal{N}_{Y/X}(\mathcal{E}\text{nd}(\mathcal{E})) \cong \mathcal{E}\text{nd}(\mathcal{N}_{Y/X}(\mathcal{E})) \quad (\forall \mathcal{E}\text{: locally free of finite rank})$$

and the monoidality of $\mathcal{N}_{Y/X}$, we obtain a well-defined homomorphism

$$\begin{array}{rccc} \text{cor}_\pi : & \text{Br}(Y) & \longrightarrow & \text{Br}(X) \\ & \Psi & & \Psi \\ & \mathcal{A} & \longmapsto & \mathcal{N}_{Y/X}(\mathcal{A}). \end{array}$$

By using Čech cohomology, we can show the commutativity of

$$\begin{tikzcd}
\mathrm{Br}(Y) \arrow[r, "\mathrm{cor}_\pi"] \arrow[d, hook, "\chi_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Br}(X) \arrow[d, hook, "\chi_X"] \\
H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \arrow[r, "\mathrm{cor}_\pi"'] & H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}).
\end{tikzcd}$$

$\square$

3. Brauer-group Mackey functor

For any profinite group G, let (fin. G-space) denote the category of finite discrete G-spaces and equivariant G-maps.

Definition 3.1. Let $\mathcal{C}$ be a Galois category, with fundamental functor F (i.e. there exists a profinite group $\pi(\mathcal{C})$ such that F gives an equivalence from $\mathcal{C}$ to (fin. $\pi(\mathcal{C})$-space)).

A cohomological Mackey functor on $\mathcal{C}$ is a pair of functors $M = (M^*, M_*)$ from $\mathcal{C}$ to Ab, where M^* is contravariant and M_* is covariant, satisfying the following conditions:
(0) $M^*(X) = M_*(X)(=: M(X)) \quad (\forall X \in \mathrm{Ob}(\mathcal{C}))$.
(1) (Additivity) For each coproduct $X \overset{i_X}{\hookrightarrow} X\coprod Y \overset{i_Y}{\hookleftarrow} Y$ in $\mathcal{C}$, the canonical morphism

$$(M^*(i_X), M^*(i_Y)) : M(X\coprod Y) \rightarrow M(X) \oplus M(Y)$$

is an isomorphism.
(2) (Mackey condition) For any pull-back diagram

$$\begin{tikzcd}
Y' \arrow[r, "\varpi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\pi'"] \\
X' \arrow[r, "\varpi"'] & X ,
\end{tikzcd}$$

the following diagram is commutative:

$$\begin{tikzcd}
M(Y) \arrow[r, "M^*(\varpi')"] \arrow[d, "M_*(\pi)"'] \arrow[dr, phantom, "\circlearrowleft"] & M(Y') \arrow[d, "M_*(\pi')"] \\
M(X) \arrow[r, "M^*(\varpi)"'] & M(X')
\end{tikzcd}$$

(3) (Cohomological condition) For any morphism $\pi : X \rightarrow Y$ in $\mathcal{C}$ with X and Y connected, we have

$$M_*(\pi) \circ M^*(\pi) = \text{multiplication by } \deg(\pi)$$

where $\deg(\pi) := \sharp F(Y)/\sharp F(X)$.

$$\begin{tikzcd}
 & M(Y) \arrow[dr, "M_*(\pi)"] \arrow[d, phantom, "\circlearrowleft"] & \\
M(X) \arrow[ur, "M^*(\pi)"] \arrow[rr, bend right, "\deg \pi"'] & {} & M(X)
\end{tikzcd}$$

A standard example is the cohomological Mackey functor on a profinite group G (in terminology of [1], a cohomological Mackey functor on the finite natural Mackey system on G):

Definition 3.2. Let G be a profinite group, and put $\mathcal{C} :=$ (fin. G-space), $F :=$ id. A cohomological Mackey functor on $\mathcal{C}$ is simply called a cohomological Mackey functor on G, and their category is denoted by $Mack_c(G)$.

Remark 3.3. Since any object X in (fin. G-space) is a direct sum of transitive G-sets of the form G/H where H is a open subgroup of G, a Mackey functor on G is equivalent to the following datum:
An abelian group $M(H)$ for each open $H \leq G$, with structure maps
- a homomorphism $\mathrm{res}^H_K : M(H) \to M(K)$ for each open $K \leq H \leq G$,
- a homomorphism $\mathrm{cor}^H_K : M(K) \to M(H)$ for each open $K \leq H \leq G$,
- a homomorphism $c_{g,H} : M(H) \to M({}^gH)$ for each open $H \leq G$ and $g \in G$,

where ${}^gH := gHg^{-1}$, satisfying certain compatibilities (cf. [1]). Here $M(G/H)$ is abbreviated to $M(H)$ for any open $H \leq G$.

Example 3.4. In this notation, for any G-module A and any $n \geq 0$, the group cohomology

$$H \mapsto H^n(H, A) \quad (\forall H \leq G \text{ open})$$

becomes a cohomological Mackey functor on G, with appropriate structure maps.

For any finite étale covering $\pi : Y \to X$, put $\mathrm{Br}^*(\pi) := \mathrm{res}_\pi$ and $\mathrm{Br}_*(\pi) := \mathrm{cor}_\pi$. Then we obtain a cohomological Mackey functor Br (and similarly Br', $H^2_{\mathrm{et}}(-, \mathbb{G}_m)$):

Theorem 3.5. *For any connected Noetherian scheme S, we have a sequence of cohomological Mackey functors on* (FEt$/S$)

$$\mathrm{Br} \hookrightarrow \mathrm{Br}' \hookrightarrow H^2_{\mathrm{et}}(-, \mathbb{G}_m).$$

Proof. We only show Mackey and cohomological conditions. Since restrictions and corestrictions are compatible with inclusions

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) \hookrightarrow H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}),$$

it suffices to show for $H^2_{\mathrm{et}}(-, \mathbb{G}_m)$.
(Mackey condition) For any pull-back diagram

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & Y' \arrow[l, "\varpi'"'] \arrow[d, "\pi'"] \\ X & X' \arrow[l, "\varpi"] \end{tikzcd}$$

in (FEt$/S$), we have a commutative diagram

$$\begin{tikzcd} \pi_*\mathbb{G}_{m,Y} \arrow[r, "\pi_*(\varpi'_\sharp)"] \arrow[dd, "N_{Y/X}"'] \arrow[ddrr, phantom, "\circlearrowright"] & \pi_*\varpi'_*\mathbb{G}_{m,Y'} \arrow[r, phantom, "\cong"] & \varpi_*\pi'_*\mathbb{G}_{m,Y'} \arrow[dd, "\varpi_*(N_{Y'/X'})"] \\ & & \\ \mathbb{G}_{m,X} \arrow[rr, "\varpi_\sharp"'] & & \varpi_*\mathbb{G}_{m,X'} \end{tikzcd}$$

This yields a commutative diagram

$$\begin{tikzcd} H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \arrow[r, "\mathrm{res}_{\varpi'}"] \arrow[d, "\mathrm{cor}_\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & H^2_{\mathrm{et}}(Y', \mathbb{G}_{m,Y'}) \arrow[d, "\mathrm{cor}_{\pi'}"] \\ H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \arrow[r, "\mathrm{res}_{\varpi}"'] & H^2_{\mathrm{et}}(X', \mathbb{G}_{m,X'}) \end{tikzcd} .$$

(Cohomological condition) For any morphism $\pi : Y \to X$ in (FEt/S) with X and Y connected, since

$$N_{Y/X} \circ \pi_\sharp : \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$$

is equal to the multiplication by $d = \deg(\pi)$

$$\begin{tikzcd} \mathbb{G}_{m,X} \arrow[r, "\pi_\sharp"] \arrow[rr, bend right, "d"'] & \pi_*\mathbb{G}_{m,Y} \arrow[r, "N_{Y/X}"] \arrow[d, phantom, "\circlearrowleft"] & \mathbb{G}_{m,X} \\ & {} & \end{tikzcd} ,$$

we obtain $\mathrm{cor}_\pi \circ \mathrm{res}_\pi = d$.

$$\begin{tikzcd} & H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \arrow[dr, "\mathrm{cor}_\pi"] & \\ H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \arrow[ur, "\mathrm{res}_\pi"] \arrow[r, "H^2_{\mathrm{et}}(\pi_\sharp)"'] \arrow[rr, bend right, "d"'] & H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}) \arrow[u, "\cong"', "\mathrm{can.}"] \arrow[r, "H^2_{\mathrm{et}}(N_{Y/X})"'] \arrow[d, phantom, "\circlearrowleft"] & H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \\ & {} & \end{tikzcd}$$

□

4. Restriction to a finite Galois covering

Thus we have obtained a Mackey functor Br on (FEt/S). By pulling back by a quasi-inverse $\mathcal{S}$ of the fundamental functor

$$F : (\mathrm{FEt}/S) \xrightarrow{\simeq} (\text{fin. } \pi(S)\text{-space}),$$

we can obtain a Mackey functor on $\pi(S)$:

Corollary 4.1. *There is a sequence of cohomological Mackey functors*

$$\mathrm{Br} \circ \mathcal{S} \hookrightarrow \mathrm{Br}' \circ \mathcal{S} \hookrightarrow H^2_{\mathrm{et}}(-, \mathbb{G}_m) \circ \mathcal{S}$$

on $\pi(S)$*, where* $\mathrm{Br} \circ \mathcal{S} := (\mathrm{Br}^* \circ \mathcal{S}, \mathrm{Br}_* \circ \mathcal{S})$ *and so on.*

Corollary 4.2. *Let X be a connected Noetherian scheme. For any finite Galois covering $\pi : Y \to X$ with $\mathrm{Gal}(Y/X) = G$, there exists a cohomological Mackey functor $\mathcal{B}\mathrm{r}$ on G which satisfies*

$$\mathcal{B}\mathrm{r}(H) \cong \mathrm{Br}(Y/H) \quad (\forall H \le G),$$

with structure maps induced from restrictions and corestrictions of Brauer groups. (We abbreviate $\mathcal{B}\mathrm{r}(G/H)$ to $\mathcal{B}\mathrm{r}(H)$.)

Proof. By the previous corollary, we have a cohomological Mackey functor $\mathrm{Br}\circ\mathcal{S}$ on $\pi(X)$. Since there is a projection $\mathrm{pr}:\pi(X)\twoheadrightarrow G^{\mathrm{op}}$, we can regard any finite G^{op}-set naturally as a finite $\pi(X)$-space, to obtain a functor

$$(\text{fin. } G^{\mathrm{op}}\text{-space})\to(\text{fin. } \pi(X)\text{-space}).$$

Pulling back by this functor, and taking the opposite Mackey functor,
we obtain

$$\begin{array}{ccccc} \mathrm{Mack}_c(\pi(X)) & \longrightarrow & \mathrm{Mack}_c(G^{\mathrm{op}}) & \xrightarrow{\mathrm{op}} & \mathrm{Mack}_c(G) \\ \cup\!\!\!\!\!\!\;\;\; & & & & \cup\!\!\!\!\!\!\;\;\; \\ M & \longmapsto & & \longrightarrow & M_G\,. \end{array}$$

In terms of subgroups of G, M_G satisfies

$$M_G(H)=M(\mathrm{pr}^{-1}(H^{\mathrm{op}}))\quad(\forall H\le G).$$

Applying this to $\mathrm{Br}\circ\mathcal{S}$, we obtain $\mathcal{B}\mathrm{r}:=(Br\circ\mathcal{S})_G\ \in\mathrm{Mack}_c(G)$. Since the equivalence $\mathcal{S}:(\text{fin. }\pi(X)\text{-space})\xrightarrow{\simeq}(\mathrm{FEt}/X)$ satisfies

$$\mathcal{S}(\pi(X)/\mathrm{pr}^{-1}(H^{\mathrm{op}}))\cong Y/H,$$

we have

$$\mathcal{B}\mathrm{r}(H)\cong\mathrm{Br}(Y/H).$$

□

Similarly we can define $\mathcal{B}\mathrm{r}'$ (and also $(H^2_{\mathrm{et}}(-,\mathbb{G}_m)\circ\mathcal{S})_G$). Since $\mathrm{Mack}_c(G)$ is an abelian category with objectwise (co-)kernels (see for example [3]), we can take the quotient Mackey functor $\mathcal{B}\mathrm{r}'\,/\,\mathcal{B}\mathrm{r}\in\mathrm{Mack}_c(G)$, which satisfies

$$(\mathcal{B}\mathrm{r}'\,/\,\mathcal{B}\mathrm{r})(H)\cong(\mathrm{Br}'(Y/H))/(\mathrm{Br}(Y/H)).$$

5. Application of Bley and Boltje's theorem

Let ℓ be a prime number. For any abelian group A, let

$$A(\ell):=\{m\in A\mid \exists e\in\mathbb{N}_{\ge 0},\ell^e m=0\}$$

be the ℓ-primary part. This is a $\mathbb{Z}_\ell$-module.

Definition 5.1 ([1])**.** For any finite group H,
H is *ℓ-hypoelementary* $\underset{\mathrm{def}}{\Leftrightarrow}$ H has a normal ℓ-subgroup with a cyclic quotient.
H is *hypoelementary* $\underset{\mathrm{def}}{\Leftrightarrow}$ H is ℓ-hypoelementary for some prime ℓ.

Fact 5.2 ([1])**.** Let M be a cohomological Mackey functor on a finite group G.
(i) Let ℓ be a prime number. If $H\le G$ is not *ℓ-hypoelementary*, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{odd}}} M(U)(\ell)^{|U|}\cong\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{even}}} M(U)(\ell)^{|U|}.$$

(ii) If $H\le G$ is not *hypoelementary* and $M(U)$ is torsion for any subgroup $U\le H$, then there is a natural isomorphism of abelian groups

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{odd}}} M(U)^{|U|}\cong\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{even}}} M(U)^{|U|}.$$

Here, $|U|$ denotes the order of U.

Applying this theorem to $\mathcal{B}\mathrm{r}$, we obtain the following relations for the Brauer groups of intermediate étale coverings:

Corollary 5.3. *Let X be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with* $\mathrm{Gal}(Y/X) = G$.
(i) *Let ℓ be a prime number. If $H \leq G$ is not ℓ-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules*

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:odd}} \mathrm{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:even}} \mathrm{Br}(Y/U)(\ell)^{|U|}.$$

(ii) *If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups*

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:odd}} \mathrm{Br}(Y/U)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:even}} \mathrm{Br}(Y/U)^{|U|}.$$

Finally, we derive some numerical equations related to Brauer groups from Corollary 5.3.

Definition 5.4. Let G be a finite group. For any subgroups $U \leq H \leq G$, put

$$\mu(U,H) := \sum_{U=H_0<\cdots<H_n=H} (-1)^n, \quad \text{Möbius function.}$$

If m (resp. m_ℓ) is an additive invariant of abelain groups (resp. $\mathbb{Z}_\ell$-modules) which is finite on Brauer groups, we obtain the following equations:

Corollary 5.5. *Let $\pi : Y \to X$ as before, $G = \mathrm{Gal}(Y/X)$.*
(i) *If $H \leq G$ is not ℓ-hypoelementary,*

$$\sum_{U \leq H} |U| \cdot \mu(U,H) \cdot m_\ell(\mathrm{Br}(Y/U)(\ell)) = 0.$$

(ii) *If $H \leq G$ is not hypoelementary,*

$$\sum_{U \leq H} |U| \cdot \mu(U,H) \cdot m(\mathrm{Br}(Y/U)) = 0.$$

For a prime ℓ and an abelian group A, its corank is defined as $\mathrm{rank}_{\mathbb{Z}_\ell}(T_\ell(A))$, where $T_\ell(A) = \varprojlim_n Ker(\ell^n : A \to A)$. In this note, we denote this by

$$\mathrm{rk}_\ell(A) := \mathrm{rank}_{\mathbb{Z}_\ell}(T_\ell(A)).$$

$\mathrm{Br}(X)(\ell)$ is known to be of finite corank, for example in the following cases ([7]):
- (C1) k: a separably closed or finite field, X: of finite type $/k$, and proper or smooth $/k$, or $\mathrm{char}(k) = 0$ or $\dim X \leq 2$.
- (C2) X: of finite type $/Spec(\mathbb{Z})$, and smooth $/Spec(\mathbb{Z})$ or proper over $\exists$open $\subset Spec(\mathbb{Z})$.

Remark that if Y/X is a finite étale covering and X satisfies (C1) or (C2), then so does Y.

Example 5.6. Assume X satisfies (C1) or (C2). For any non-ℓ-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U|\mu(U,H) \cdot \mathrm{rk}_\ell(\mathrm{Br}(Y/H)(\ell)) = 0.$$

Another example is related with the comparison of Br and Br′. By Gabber's lemma, for any finite étale covering Y/X, we have

$$\mathrm{Br}'(X)/\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(Y)/\mathrm{Br}(Y).$$

In particular, if $\mathrm{Br}(Y) \subset \mathrm{Br}(Y)'$ is of finite index, then so is $\mathrm{Br}(X) \subset \mathrm{Br}(X)'$.

Example 5.7. Assume X satisfies $[\mathrm{Br}'(Y) : \mathrm{Br}(Y)] < \infty$. Then for any non-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U|\mu(U, H) \cdot [\mathrm{Br}'(Y/U) : \mathrm{Br}(Y/U)] = 0.$$

References

[1] W. Bley, R. Boltje, Cohomological Mackey functors in number theory, J. Number Theory 105 (2004), 1-37.

[2] R. Boltje, Class group relations from Burnside ring idempotents, J. Number Theory 66 (1997), 291-305.

[3] S. Bouc, Green functors and G-sets, Lecture Notes in Mathematics, 1671. Springer-Verlag, Berlin, 1997.

[4] D. Ferrand, Un foncteur norme, Bull. Soc. math. France 126 (1998), 1-49.

[5] T. J. Ford, Hecke actions on Brauer groups, J. Pure Appl. Algebra 33 (1984), 11-17.

[6] O. Gabber, Some theorems on Azumaya algebras, Lecture Notes in Math. 844 Springer-Verlag (1980), 129-210.

[7] A. Grothendieck, Le groupe de Brauer I,II,III, In Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, (1968), 46-188.

[8] M.-A. Knus, M. Ojanguren, A Norm for modules and algebras, Math. Z. 142 (1975), 33-45.

[9] J. P. Murre, Lectures on an introduction to Grothendieck's theory of the fundamental group, Tata Institute of Fundamental Research Lectures on Mathmatics, No 40 Tata Institute of Fundamental Research, Bombay, 1967.

Graduate School of Mathematical Sciences, The University of Tokyo 3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan

E-mail address: `deutsche@ms.u-tokyo.ac.jp`