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<td>Author(s)</td>
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<td>Citation</td>
<td>代数幾何学シンポジウム記録</td>
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<tr>
<td>Issue Date</td>
<td>2007</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/214843">http://hdl.handle.net/2433/214843</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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MACKEY-FUNCTOR STRUCTURE ON THE BRAUER GROUPS
OF A FINITE GALOIS COVERING OF SCHEMES

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Abstract. For any finite étale covering of schemes, we can associate two homomorphisms for Brauer groups, namely the pull-back and the norm map. These homomorphisms make Brauer groups into a bivariant functor (a Mackey functor). Restricting to a finite Galois covering of schemes, we obtain a cohomological Mackey functor on its Galois group. This is a generalization of the result for rings by Ford [5]. Applying Bley and Boltje’s theorem [1], we can derive certain isomorphisms for the Brauer groups of intermediate coverings.

1. Introduction

In this paper, a scheme $S$ is always assumed to be Noetherian, and $\pi(S)$ denotes its étale fundamental group. Since we use Čech cohomology, we assume $S$ satisfies the following:

Assumption 1.1. For any finite set $E$ of points of $S$, there exists an open set $U \subset S$, such that $U$ contains every point in $E$.

As for the étale fundamental group and related notion, we follow the terminology in [9]. For example a finite étale covering is just a finite étale morphism of schemes.

Our aim is to make the following generalization of the result for rings by Ford [5].

Corollary (Corollary 4.2). Let $\pi : Y \to X$ be a finite Galois covering of schemes with Galois group $G$. Then the correspondence

$$H \leq G \mapsto \text{Br}(Y/H)$$

forms a cohomological Mackey functor on $G$.

This follows from our main theorem;

Theorem (Theorem 3.5). Let $S$ be a connected Noetherian scheme. Let $(\text{F}\text{E}t/S)$ denote the category of finite étale coverings over $S$. Then, the Brauer group functor $\text{Br}$ forms a cohomological Mackey functor on $(\text{F}\text{E}t/S)$.

As in Definition 3.1, a Mackey functor is a bivariant pair of functors $\text{Br} = (\text{Br}^*, \text{Br}_*)$. For any morphism $\pi : Y \to X$, the contravariant part $\text{Br}^*(\pi) : \text{Br}(X) \to \text{Br}(Y)$ is the pull-back, and $\text{Br}_*(\pi) : \text{Br}(Y) \to \text{Br}(X)$ is the norm map defined later.

By applying Bley and Boltje’s theorem (Fact 5.2) to Corollary 4.2, we can obtain certain relations between Brauer groups of intermediate coverings:

The author wishes to thank Professor Kazuhiro Fujiwara for his useful comments.
Corollary (Corollary 5.3). Let $X$ be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with $\text{Gal}(Y/X) = G$.

(i) Let $\ell$ be a prime number. If $H \leq G$ is not $\ell$-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules
\[
\bigoplus_{U = H_0 < \cdots < H_n = H \atop \text{n. odd}} \text{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{U = H_0 < \cdots < H_n = H \atop \text{n. even}} \text{Br}(Y/U)(\ell)^{|U|}.
\]

(ii) If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups
\[
\bigoplus_{U = H_0 < \cdots < H_n = H \atop \text{n. odd}} \text{Br}(Y/U)^{|U|} \cong \bigoplus_{U = H_0 < \cdots < H_n = H \atop \text{n. even}} \text{Br}(Y/U)^{|U|}.
\]

2. Restriction and corestriction

Remark 2.1. For any scheme $X$, there exists a natural monomorphism
\[
\chi_X : \text{Br}(X) \hookrightarrow \text{Br}'(X) := H^2_{\text{et}}(X, \mathbb{G}_m, X)_{\text{tor}},
\]
such that for any morphism $\pi : Y \to X$,
\[
\begin{array}{ccc}
\text{Br}(X) & \xrightarrow{\pi^*} & \text{Br}(Y) \\
\text{Br}^2(X, \mathbb{G}_m, X) & \xrightarrow{\chi_X} & \text{Br}^2(Y, \mathbb{G}_m, Y)
\end{array}
\]
is a commutative diagram.

Here $\pi^* : \text{Br}(X) \to \text{Br}(Y)$ is the pull-back of Azumaya algebras, while $\pi^* : H^2_{\text{et}}(X, \mathbb{G}_m, X) \to H^2_{\text{et}}(Y, \mathbb{G}_m, Y)$ is defined as the composition of the canonical morphism
\[
H^2_{\text{et}}(X, \mathbb{G}_m, X) \to H^2_{\text{et}}(Y, \mathbb{G}_m, Y)
\]
and
\[
H^2_{\text{et}}(\pi_* : H^2_{\text{et}}(X, \mathbb{G}_m, X) \to H^2_{\text{et}}(X, \pi_* \mathbb{G}_m, X),
\]
where $\pi_* : \mathbb{G}_m, X \to \pi_* \mathbb{G}_m, Y$ is the canonical (structural) homomorphism of étale sheaves on $X$. We call these $\pi^*$ the restriction maps.

Remark 2.2. For any finite étale covering $\pi : Y \to X$, there exists a homomorphism of étale sheaves on $X$
\[
N_{Y/X} : \pi_* \mathbb{G}_m, Y \to \mathbb{G}_m, X
\]
which induces the norm map for finite étale ring extensions.

When $\pi : Y \to X$ is a finite étale covering, the canonical homomorphism $H^2_{\text{et}}(X, \pi_* \mathbb{G}_m, Y) \to H^2_{\text{et}}(Y, \mathbb{G}_m, Y)$ becomes isomorphic (cf. [6]). By composing $H^2_{\text{et}}(N_{Y/X})$ with the inverse of this canonical isomorphism, we define the corestriction map for cohomology groups:
\[
\text{cor}_\pi : H^2_{\text{et}}(Y, \mathbb{G}_m, Y) \xrightarrow{\cong} H^2_{\text{et}}(X, \pi_* \mathbb{G}_m, Y) \xrightarrow{H^2_{\text{et}}(N_{Y/X})^{-1}} H^2_{\text{et}}(X, \mathbb{G}_m, X).
\]
Proposition 2.3. Let \( \pi : Y \to X \) as before. There exists a corestriction homomorphism for Brauer groups

\[
\text{cor}_\pi : \text{Br}(Y) \to \text{Br}(X),
\]

such that

\[
\begin{array}{ccc}
\text{Br}(Y) & \xrightarrow{\text{cor}_\pi} & \text{Br}(X) \\
\downarrow_{\chi_Y} & & \downarrow_{\chi_X} \\
H^2_{\text{et}}(Y, \mathbb{G}_m,Y) & \xrightarrow{\text{cor}_\pi} & H^2_{\text{et}}(X, \mathbb{G}_m,X)
\end{array}
\]

is commutative.

To construct \( \text{cor} : \text{Br}(Y) \to \text{Br}(X) \), we define a monoidal functor

\( N_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X) \).

Lemma 2.4. Let \( \pi : Y \to X \) be a finite étale covering of constant degree \( d \). There exists a monoidal functor (unique up to a natural isomorphism)

\[ N_\pi = N_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X), \]

\( N_\pi \) is defined by as in (i).

Remark 2.5. For any finite étale covering \( \pi : Y \to X \) of constant degree \( d \), there exists a fppf morphism \( f : X' \to X \) such that \( Y \times_X X' \) is isomorphic to a disjoint union of \( d \)-copies of \( X' \).

Proof. When \( Y \) is isomorphic to a disjoint union of \( d \)-copies of \( X \), then \( N_{Y/X} \) is defined as in (i).

For a general case, remark that

\[
\begin{array}{ccc}
Y' & \xrightarrow{\pi'} & X' \\
g \downarrow & & \downarrow f \\
Y & \xrightarrow{\pi} & X
\end{array}
\]

there exists a natural isomorphism of monoidal functors

\[ N_{Y'/X'} \circ g^* \cong f^* \circ N_{Y/X}. \]

Proof. When \( Y \) is isomorphic to a disjoint union of \( d \)-copies of \( X \), then \( N_{Y/X} \) is defined as in (i).

For a general case, remark that

\[
\begin{array}{ccc}
Y' & \cong \coprod_d X' & \xrightarrow{\pi'} & X' \\
g \downarrow & & \downarrow f \\
Y & \xrightarrow{\pi} & X
\end{array}
\]

For any \( \mathcal{F} \in \text{q-Coh}(Y) \), put \( \tilde{\mathcal{F}} := N_{Y'/X'}(g^*(\mathcal{F})) \). Then \( \tilde{\mathcal{F}} \) descends to yield \( N_{Y/X}(\mathcal{F}) \in \text{q-Coh}(X) \). Thus we obtain a monoidal functor \( N_{Y/X} \). This construction does not depend on the choice of \( f \), up to an isomorphism of monoidal functors. By the reduction to the disjoint-union case as above, we can show (ii).
While this $N_{Y/X}$ is a generalization of the norm functor for a finite étale ring extension (Knus-Ojanguren [8], Ferrand [4]), it is also possible to define $N_{Y/X}$ by gluing those for affines.

**Lemma 2.6.** Let $\pi : Y \rightarrow X$ be a finite étale covering of constant degree $d$. $N_{Y/X}$ has the following properties:

(0) $N_{Y/X}$ is monoidal.

(1) For any $F, G \in q\text{-Coh}(Y)$, there exists a functorial morphism

$$\theta_{Y/X} : N_{Y/X}(\text{Hom}_{O_Y}(F, G)) \rightarrow \text{Hom}_{O_X}(N_{Y/X}(F), N_{Y/X}(G)).$$

(1') Moreover if $G$ is locally free of finite rank, this is an isomorphism.

(2) There exists a natural isomorphism

$$N_{Y/X}(O_Y^\oplus n) \cong O_X^\oplus nd.$$

(2') More generally, if $F$ is locally free $O_Y$-module of finite rank $n$, then $N_{Y/X}(F)$ becomes locally free $O_X$-module of rank $nd$.

For a general (non-constant degree) $\pi : Y \rightarrow X$, we can define the norm functor on each connected component of $X$ as above, and glue them to obtain the norm functor $N_{Y/X} : q\text{-Coh}(Y) \rightarrow q\text{-Coh}(X)$.

**Proof.** Conditions (0) and (2) follow from the definition of $N_{Y/X}$. By taking an affine cover $X = \bigcup_{i \in I} U_i$, (2') reduces to the case where $X, Y$ are affine, shown by Ferrand [4]. As for condition (1), existence of $\theta_{Y/X}$ simply follows from the fact that $N_{Y/X}$ is a monoidal functor between closed symmetric monoidal categories.

(1') is shown by a reduction to the affine case. \(\square\)

**Proof.** (proof of Proposition) By the above lemma, especially we have an isomorphism

$$N_{Y/X}(M_n(O_Y)) \cong M_{nd}(O_X)$$

of $O_X$-algebras, for any finite étale covering $Y/X$ of constant degree $d$.

Remark that for any $O_Y$-algebra $A$ of finite type, $A$ is an Azumaya algebra if and only if $A$ is étale locally isomorphic to $M_n(O_Y)$. Thus for any Azumaya algebra $A$, there exists a covering $V := \{V_i \xrightarrow{g_i} Y\}_{i \in I}$ of $Y$ in the étale topology (simply written $\mathcal{V} \in \text{Cov}_{\text{et}}(Y)$) such that

$$g_i^* A \cong M_{n_i}(O_{V_i}) \quad (\exists n_i \in \mathbb{N}).$$

Replacing $V$ by its refinement, we may assume that there exists a covering $U = \{U_i \xrightarrow{f_i} X\}_{i \in I} \in \text{Cov}_{\text{et}}(X)$ such that

$$\mathcal{V} = \pi^* \mathcal{U} := \{Y \times_X U_i \xrightarrow{pr} Y\}_{i \in I}.$$

Then we have $f_i^* N_{Y/X}(A) \cong N_{U_i/X}(g_i^* A) \cong M_{n_i}(O_{U_i})$. Thus $N_{Y/X}(A)$ also becomes an Azumaya algebra.

By the isomorphism

$$N_{Y/X}(\mathcal{E}(\text{End}(\mathcal{E}))) \cong \text{End}(N_{Y/X}(\mathcal{E})) \quad (\forall \mathcal{E} : \text{locally free of finite rank})$$

and the monoidality of $N_{Y/X}$, we obtain a well-defined homomorphism

$$\text{cor}_{\pi} : Br(Y) \xrightarrow{\psi} Br(X) \xrightarrow{\psi} N_{Y/X}(A).$$
By using Čech cohomology, we can show the commutativity of
\[
\begin{array}{c}
\text{Br}(Y) \xrightarrow{\text{cor}_\pi} \text{Br}(X) \\
\xrightarrow{\chi_Y} \circ \\
H_2^\text{et}(Y, \mathbb{G}_m, Y) \xrightarrow{\text{cor}_\pi} H_2^\text{et}(X, \mathbb{G}_m, X).
\end{array}
\]

\[\Box\]

3. BRAUER-GROUP MACKEY FUNCTOR

For any profinite group \(G\), let (fin. \(G\)-space) denote the category of finite discrete \(G\)-spaces and equivariant \(G\)-maps.

**Definition 3.1.** Let \(\mathcal{C}\) be a Galois category, with fundamental functor \(F\) (i.e. there exists a profinite group \(\pi(\mathcal{C})\) such that \(F\) gives an equivalence from \(\mathcal{C}\) to (fin. \(\pi(\mathcal{C})\)-space)).

A cohomological Mackey functor on \(\mathcal{C}\) is a pair of functors \(M = (M^*, M_*): \mathcal{C} \to \text{Ab}\), where \(M^*\) is contravariant and \(M_*\) is covariant, satisfying the following conditions:

0. \(M^*(X) = M_*(X)(= M(X))\) \((\forall X \in \text{Ob}(\mathcal{C}))\).

1. (Additivity) For each coproduct \(X \xleftarrow{i_X} \coprod Y \xrightarrow{i_Y} Y\) in \(\mathcal{C}\), the canonical morphism
\[
(M^*(i_X), M^*(i_Y)): M(X \coprod Y) \to M(X) \oplus M(Y)
\]
is an isomorphism.

2. (Mackey condition) For any pull-back diagram
\[
\begin{array}{ccc}
Y' & \xrightarrow{\pi'} & Y \\
\downarrow{\pi} & \Box & \downarrow{\pi'} \\
X' & \xrightarrow{\pi} & X,
\end{array}
\]
the following diagram is commutative:

\[
\begin{array}{ccc}
M(Y) & \xrightarrow{M^*(\pi')} & M(Y') \\
\downarrow{M_*(\pi)} & \Box & \downarrow{M_*(\pi')} \\
M(X) & \xrightarrow{M^*(\pi)} & M(X')
\end{array}
\]

3. (Cohomological condition) For any morphism \(\pi: X \to Y\) in \(\mathcal{C}\) with \(X\) and \(Y\) connected, we have
\[
M_*(\pi) \circ M^*(\pi) = \text{multiplication by } \deg(\pi)
\]
where \(\deg(\pi) := \sharp F(Y)/\sharp F(X)\).
A standard example is the cohomological Mackey functor on a profinite group $G$ (in terminology of [1], a cohomological Mackey functor on the finite natural Mackey system on $G$):

**Definition 3.2.** Let $G$ be a profinite group, and put $C := \text{fin.} \ G$-space, $F := \text{id.}$ A cohomological Mackey functor on $C$ is simply called a cohomological Mackey functor on $G$, and their category is denoted by $\text{Mack}_c(G)$.

**Remark 3.3.** Since any object $X$ in $\text{fin.} \ G$-space is a direct sum of transitive $G$-sets of the form $G/H$ where $H$ is a open subgroup of $G$, a Mackey functor on $G$ is equivalent to the following datum:

An abelian group $M(H)$ for each open $H \leq G$, with structure maps
- a homomorphism $\text{res}^H_K : M(H) \to M(K)$ for each open $K \leq H \leq G$,
- a homomorphism $\text{cor}^H_K : M(K) \to M(H)$ for each open $K \leq H \leq G$,
- a homomorphism $\text{c}^g_H : M(H) \to M(\text{g}H)$ for each open $H \leq G$ and $g \in G$,

where $\text{g}H := gHg^{-1}$, satisfying certain compatibilities (cf. [1]). Here $M(G/H)$ is abbreviated to $M(H)$ for any open $H \leq G$.

**Example 3.4.** In this notation, for any $G$-module $A$ and any $n \geq 0$, the group cohomology

$$H \hookrightarrow H^n(H, A) \quad (\forall H \leq G \text{ open})$$

becomes a cohomological Mackey functor on $G$, with appropriate structure maps.

For any finite étale covering $\pi : Y \to X$, put $\text{Br}^*(\pi) := \text{res}_\pi$ and $\text{Br}^*(\pi) := \text{cor}_\pi$. Then we obtain a cohomological Mackey functor $\text{Br}$ (and similarly $\text{Br}'$, $H^2_{\text{et}}(-, \mathbb{G}_m)$):

**Theorem 3.5.** For any connected Noetherian scheme $S$, we have a sequence of cohomological Mackey functors on $(\text{FEt}/S)$

$$\text{Br} \hookrightarrow \text{Br}' \hookrightarrow H^2_{\text{et}}(-, \mathbb{G}_m).$$

**Proof.** We only show Mackey and cohomological conditions. Since restrictions and corestrictions are compatible with inclusions

$$\text{Br}(X) \hookrightarrow \text{Br}'(X) \hookrightarrow H^2_{\text{et}}(X, \mathbb{G}_m, X),$$

it suffices to show for $H^2_{\text{et}}(-, \mathbb{G}_m)$.

(Mackey condition) For any pull-back diagram

$$\begin{array}{ccc}
Y' & \to & Y \\
\downarrow & & \downarrow \\
X' & \to & X
\end{array}$$

in $(\text{FEt}/S)$, we have a commutative diagram

$$\begin{array}{ccc}
\pi_+ \mathbb{G}_m, Y & \to & \pi_+ \mathbb{G}_m, Y' \\
\downarrow & & \downarrow \\
\mathbb{G}_m, X & \to & \mathbb{G}_m, X'
\end{array}$$

where

$$N_{Y'/X} \hookrightarrow \mathbb{G}_m, Y'$$

and

$$\mathbb{G}_m, X \to \mathbb{G}_m, X'.$$
This yields a commutative diagram

\[
\begin{array}{ccc}
H^2_{\text{et}}(Y, \mathbb{G}_m, Y) & \xrightarrow{\text{res}_{\pi'}} & H^2_{\text{et}}(Y', \mathbb{G}_m, Y') \\
\downarrow \text{cor}_\pi & \circ & \downarrow \text{cor}_{\pi'} \\
H^2_{\text{et}}(X, \mathbb{G}_m, X) & \xrightarrow{\text{res}_{\pi}} & H^2_{\text{et}}(X', \mathbb{G}_m, X') .
\end{array}
\]

(Cohomological condition) For any morphism \( \pi : Y \to X \) in \((\text{FEt}/S)\) with \( X \) and \( Y \) connected, since

\[ N_{Y/X} \circ \pi_2 : \mathbb{G}_m, X \to \mathbb{G}_m, X \]

is equal to the multiplication by \( d = \deg(\pi) \)

we obtain \( \text{cor}_\pi \circ \text{res}_\pi = d \).

Thus we have obtained a Mackey functor \( \text{Br} \) on \((\text{FEt}/S)\). By pulling back by a quasi-inverse \( S \) of the fundamental functor

\[ F : (\text{FEt}/S) \xrightarrow{\sim} (\text{fin. } \pi(S)-\text{space}), \]

we can obtain a Mackey functor on \( \pi(S) \):

**Corollary 4.1.** There is a sequence of cohomological Mackey functors

\[ \text{Br} \circ S \hookrightarrow \text{Br}' \circ S \hookrightarrow H^2_{\text{et}}(-, \mathbb{G}_m) \circ S \]

on \( \pi(S) \), where \( \text{Br} \circ S := (\text{Br}^* \circ S, \text{Br}_s \circ S) \) and so on.

**Corollary 4.2.** Let \( X \) be a connected Noetherian scheme. For any finite Galois covering \( \pi : Y \to X \) with \( \text{Gal}(Y/X) = G \), there exists a cohomological Mackey functor \( \text{Br} \) on \( G \) which satisfies

\[ \text{Br}(H) \cong \text{Br}(Y/H) \quad (\forall H \leq G), \]

with structure maps induced from restrictions and corestrictions of Brauer groups. (We abbreviate \( \text{Br}(G/H) \) to \( \text{Br}(H) \).)
Proof. By the previous corollary, we have a cohomological Mackey functor $Br \circ S$ on $\pi(X)$. Since there is a projection $\text{pr} : \pi(X) \to G^{\text{op}}$, we can regard any finite $G^{\text{op}}$-set naturally as a finite $\pi(X)$-space, to obtain a functor

$$\text{fin. } G^{\text{op}}\text{-space} \to \text{fin. } \pi(X)\text{-space}.$$ 

Pulling back by this functor, and taking the opposite Mackey functor, we obtain

$$\text{Mack}_c(\pi(X)) \to \text{Mack}_c(G)^{\text{op}} \to \text{Mack}_c(G).$$

In terms of subgroups of $G$, $M_G$ satisfies

$$M_G(H) = M(\text{pr}^{-1}(H^{\text{op}})) \quad (\forall H \leq G).$$

Applying this to $Br \circ S$, we obtain $Br := (Br \circ S)_G \in \text{Mack}_c(G)$. Since the equivalence $S : (\text{fin. } \pi(X)\text{-space}) \to (\text{FEt}/X)$ satisfies

$$S(\pi(X)/\text{pr}^{-1}(H^{\text{op}})) \cong Y/H,$$

we have

$$Br(H) \cong Br(Y/H).$$

Similarly we can define $Br'$ (and also $(H_2^{\text{et}}(-, \mathbb{G}_m) \circ S)_G$). Since $\text{Mack}_c(G)$ is an abelian category with objectwise (co-)kernels (see for example [3]), we can take the quotient Mackey functor $Br'/Br \in \text{Mack}_c(G)$, which satisfies

$$(Br'/Br)(H) \cong (Br'(Y/H))/(Br(Y/H)).$$

5. Application of Bley and Boltje’s theorem

Let $\ell$ be a prime number. For any abelian group $A$, let

$$A(\ell) := \{ m \in A \mid \exists e \in \mathbb{N}_{\geq 0}, \ell^e m = 0 \}$$

be the $\ell$-primary part. This is a $\mathbb{Z}_\ell$-module.

**Definition 5.1 ([1]).** For any finite group $H$, $H$ is $\ell$-hypoelementary if $H$ has a normal $\ell$-subgroup with a cyclic quotient.

$H$ is hypoelementary if $H$ is $\ell$-hypoelementary for some prime $\ell$.

**Fact 5.2 ([1]).** Let $M$ be a cohomological Mackey functor on a finite group $G$.

(i) Let $\ell$ be a prime number. If $H \leq G$ is not $\ell$-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{U=H_0 < \cdots < H_n = H}^{n \text{ odd}} M(U)(\ell)^{|U|} \cong \bigoplus_{U=H_0 < \cdots < H_n = H}^{n \text{ even}} M(U)(\ell)^{|U|}.$$ 

(ii) If $H \leq G$ is not hypoelementary and $M(U)$ is torsion for any subgroup $U \leq H$, then there is a natural isomorphism of abelian groups

$$\bigoplus_{U=H_0 < \cdots < H_n = H}^{n \text{ odd}} M(U)^{|U|} \cong \bigoplus_{U=H_0 < \cdots < H_n = H}^{n \text{ even}} M(U)^{|U|}.$$ 

Here, $|U|$ denotes the order of $U$. 

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Applying this theorem to $\text{Br}$, we obtain the following relations for the Brauer groups of intermediate étale coverings:

**Corollary 5.3.** Let $X$ be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with $\text{Gal}(Y/X) = G$.

(i) Let $\ell$ be a prime number. If $H \leq G$ is not $\ell$-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)(\ell)^{|U|}.$$ 

(ii) If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups

$$\bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)^{|U|} \cong \bigoplus_{U = H_0 < \cdots < H_n = H} \text{Br}(Y/U)^{|U|}.$$ 

Finally, we derive some numerical equations related to Brauer groups from Corollary 5.3.

**Definition 5.4.** Let $G$ be a finite group. For any subgroups $U \leq H \leq G$, put

$$\mu(U, H) := \sum_{U = H_0 < \cdots < H_n = H} (-1)^n, \quad \text{Möbius function}.$$ 

If $m$ (resp. $m_\ell$) is an additive invariant of abelian groups (resp. $\mathbb{Z}_\ell$-modules) which is finite on Brauer groups, we obtain the following equations:

**Corollary 5.5.** Let $\pi : Y \to X$ as before, $G = \text{Gal}(Y/X)$.

(i) If $H \leq G$ is not $\ell$-hypoelementary,

$$\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m_\ell(\text{Br}(Y/U)(\ell)) = 0.$$ 

(ii) If $H \leq G$ is not hypoelementary,

$$\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m(\text{Br}(Y/U)) = 0.$$ 

For a prime $\ell$ and an abelian group $A$, its corank is defined as $\text{rank}_{\mathbb{Z}_\ell}(T_\ell(A))$, where $T_\ell(A) = \lim_{\leftarrow n} \text{Ker}(\ell^n : A \to A)$. In this note, we denote this by

$$\text{rk}_\ell(A) := \text{rank}_{\mathbb{Z}_\ell}(T_\ell(A)).$$

$\text{Br}(X)(\ell)$ is known to be of finite corank, for example in the following cases ([7]):

- (C1) $k$: a separably closed or finite field, $X$: of finite type $/k$, and proper or smooth $/k$, or char($k$) = 0 or dim $X \leq 2$.
- (C2) $X$: of finite type $/\text{Spec}(\mathbb{Z})$, and smooth $/\text{Spec}(\mathbb{Z})$ or proper over $\exists \text{ open } \subset \text{Spec}(\mathbb{Z})$.

Remark that if $Y/X$ is a finite étale covering and $X$ satisfies (C1) or (C2), then so does $Y$.

**Example 5.6.** Assume $X$ satisfies (C1) or (C2). For any non-$\ell$-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U| \mu(U, H) \cdot \text{rk}_\ell(\text{Br}(Y/H)(\ell)) = 0.$$
Another example is related with the comparison of $\text{Br}$ and $\text{Br}'$. By Gabber’s lemma, for any finite étale covering $Y/X$, we have

$$\text{Br}'(X)/\text{Br}(X) \hookrightarrow \text{Br}'(Y)/\text{Br}(Y).$$

In particular, if $\text{Br}(Y) \subset \text{Br}(Y)'$ is of finite index, then so is $\text{Br}(X) \subset \text{Br}(X)'.$

**Example 5.7.** Assume $X$ satisfies $[\text{Br}'(Y) : \text{Br}(Y)] < \infty$. Then for any non-hypoelementary subgroup $H \leq G$, we have an equation

$$
\sum_{U \leq H} |U| \mu(U, H) \cdot [\text{Br}'(Y/U) : \text{Br}(Y/U)] = 0.
$$

**References**


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