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# Universal liftings of chain complexes based on non-commutative parameter algebras

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## 1 Introduction

Throughout this article, $k$ always denotes a field and $R$ denotes an associative $k$-algebra. By an $R$-module, we always mean a left module.

For a given $R$-module $M$, we are mainly concerned with the deformation of $M$ with fixing $R$.

Let $\mathcal{C}_k$ be the category of commutative artinian local $k$-algebras with residue field $k$ and $k$-algebra homomorphisms. We consider the covariant functor

$$\mathcal{F}_M : \mathcal{C}_k \to (Sets),$$

which maps $A \in \mathcal{C}_k$ to the set of infinitesimal deformations of $M$ along $A$, i.e.

$$\mathcal{F}_M(A) = \left\{ \begin{array}{c} (R,A)\text{-bimodules } X \text{ that are flat over } A \\ \text{and } X \otimes^L_A k \cong M \text{ as left } R\text{-modules} \end{array} \right\} / \simeq$$

where $\simeq$ means $(R,A)$-bimodule isomorphism.

Under these circumstances the following theorem is known. (cf. [1], [2], [3].)

**Theorem 1.1 (Schlessinger's Theorem 1968)** *Suppose* $\mathrm{Ext}^1_R(M,M)$ *is of finite dimension as a* $k$*-vector space. Then the functor* $\mathcal{F}_M$ *is pro-representable.*

*More precisely, there exist a commutative noetherian complete local* $k$*-algebra* $Q$ *with residue field* $k$ *and an* $(R,Q)$*-bimodule* $U$ *that is flat over* $Q$ *such that there is an isomorphism*

$$\mathrm{Hom}_{k-alg}(Q,\ \ ) \cong \mathcal{F}_M$$

*as functors on* $\mathcal{C}_k$*. The isomorphism is given in such a way that a* $k$*-algebra map* $f : Q \to A$ *is mapped to* $[U \otimes_{Q\ f} A] \in \mathcal{F}_M(A)$ *for* $A \in \mathcal{C}_k$*.*

We call $U$ the universal family of deformations of $M$, and $Q$ (or Spec$Q$) the commutative parameter algebra (resp. parameter space) of the universal family $U$.

Under the setting of Theorem 1.1, since $U$ is flat as a right $Q$-module, we have a functor between derived categories:

$$U \otimes_Q^L - \quad : D(Q) \to D(R).$$

Noting that $U \otimes_Q^L k = M$, we see the functor induces an algebra map between Yoneda algebras:

$$\rho^\bullet \ : \ \mathrm{Ext}_Q^\bullet(k,k) \to \mathrm{Ext}_R^\bullet(M,M).$$

Of most interest is the map

$$\rho^2 \ : \ \mathrm{Ext}_Q^2(k,k) \to \mathrm{Ext}_R^2(M,M),$$

which is often called the obstruction map.

We should notice that this obstruction map is not a good one as a comparison of cohomology modules. We show this by the following elementary example.

EXAMPLE 1.2 Consider the indecomposable Jordan canonical form of an $n \times n$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Setting $R = k[x]$, we know that this is equivalent to considering an $R$-module $M = k[x]/(x^n)$. In this case, we can take $Q = k[[t_0, \dots, t_{n-1}]]$ as the parameter algebra for the universal deformation of $M$ and $U = Q[x]/(x^n + t_{n-1}x^{n-1} + \cdots + t_0)$ as the universal deformation. This gives the so-called Sylvester family of matrices.

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 \\ -t_0 & -t_1 & -t_2 & \cdots & -t_{n-1} \end{pmatrix}$$

In this example, we have

$$\begin{array}{rl} \mathrm{Ext}_Q^2(k,k) & = \langle \text{Koszul relations of } t_i \text{ of degree } 2 \rangle^* \\ & \downarrow \ \rho^2 \\ \mathrm{Ext}_R^2(M, & M) = (0) \end{array}$$

Compared with that $\mathrm{Ext}_R^2(M,M) = (0)$, $\mathrm{Ext}_Q^2(k,k)$ is rather a large space with dimension $\binom{n}{2}$. Notice that the Koszul relations of degree 2 are derived from commutativity relations of the parameters $t_0, \dots, t_{n-1}$. If we take those parameters $t_0, \dots, t_{n-1}$ as non-commutative variables, then we would have better comparison map as the obstruction map. This is our first main idea of this article.

(1) Parameter algebras should be non-commutative.

The second idea is the following:

(2) We consider chain complexes instead of modules.

This is to avoid any argument concerning the flatness. The reason for this is that the local criterion of flatness does not necessarily hold for modules over non-commutative rings. A deformation of a chain complex is nothing but a lifting of a chain complex.

Consequently we naturally come to think of the liftings of chain complexes based on non-commutative parameter algebras.

## 2 Existence and uniqueness of universal liftings

First we introduce the notation for chain complexes. We say $\mathbb{F} = (F, d)$ is a chain complex of $R$-modules if $F = \oplus_{i \in \mathbb{Z}} F_i$ is a graded $R$-module and $d : F \longrightarrow F[-1]$ is a graded $R$-homomorphism with $d^2 = 0$. A chain complex $\mathbb{F} = (F, d)$ is said to be projective if the underlying graded $R$-module $F$ is projective. Let $f : F \longrightarrow F[i]$, $g : F \longrightarrow F[j]$ be graded homomorphisms of graded $R$-modules. We define

$$[f, g] := fg + (-1)^{i+j} gf \quad : \quad F \longrightarrow F[i+j].$$

We call $f : F \longrightarrow F[i]$ a chain homomorphism whenever it satisfies $[d, f] = 0$. We denote by $f \sim 0$ if there is a graded homomorphism $h : F \longrightarrow F[i+1]$ with $f = [d, h]$. If $\mathbb{F} = (F, d)$ is a projective complex, we define

$$\mathrm{Ext}^i_R(\mathbb{F}, \mathbb{F}) \overset{def}{=} \{\text{chain homomorphisms } F \longrightarrow F[i]\}/\sim.$$

Let $\varphi : R \longrightarrow S$ be a $k$-algebra map. We define

$$(F, d) \otimes_R {}_\varphi S \overset{def}{=} (F \otimes_R {}_\varphi S, d \otimes_R {}_\varphi S),$$

where ${}_\varphi S$ is a right $S$-module $S$ regarded as a left $R$-module via $\varphi$.

We introduce the category $\mathcal{A}_k$ which is the basis of deformation.

- The objects of the category $\mathcal{A}_k$ are artinian local $k$-algebras $A$ with $A/\mathfrak{m}_A = k$, where $\mathfrak{m}_A$ denotes the Jacobson radical of $A$. (They are not necessarily commutative, but they are finite dimensional $k$-algebras.)
- Morphisms of $\mathcal{A}_k$ are $k$-algebra homomorphisms.

**Definition 2.1** Let $A \in \mathcal{A}_k$ and let $\mathbb{F} = (F, d)$ be a projective complex of $R$-modules. We say that $(F \otimes_k A, \Delta)$ is a lifting of $\mathbb{F}$ to $A$ if the following conditions hold:

$$\begin{cases} (F \otimes_k A, \Delta) \text{ is a chian complex of } R \otimes_k A^{op}\text{-modules, and} \\ (F \otimes_k A, \Delta) \otimes_A k = (F, d) \end{cases}$$

Fixing a projective complex $\mathbb{F} = (F, d)$ of $R$-modules, we define the functor

$$\mathcal{F} : \mathcal{A}_k \to (Sets)$$

as follows:

$$\mathcal{F}(A) := \frac{\{\text{liftings of } \mathbb{F} \text{ to } A\}}{< \text{chain isomorphisms of } R \otimes_k A^{op}\text{-modules} >} \qquad (A \in \mathcal{A}_k).$$

The following is one of the main results of this talk.

**Theorem 2.2** *Suppose that a projective complex $\mathbb{F} = (F, d)$ of $R$-modules satisfies $r = \dim_k \operatorname{Ext}^1_R(\mathbb{F}, \mathbb{F}) < \infty$. Then the functor $\mathcal{F}$ is pro-representable. I.e. There exist a (non-commutative) complete local $k$-algebra $P_0$ and a lifting chain complex $\mathbb{L}_0 = (F \widehat{\otimes}_k P_0, \Delta_0)$ of $\mathbb{F}$ to $P_0$ satisfying*

$$\mathcal{F} \cong \operatorname{Hom}_{k\text{-}alg}(P_0, \quad )$$

*as functors on $\mathcal{A}_k$. The isomorphism is given in such a way that a $k$-algebra map $f : P_0 \to A$ is mapped to $[\mathbb{L} \otimes_{P_0\, f} A] \in \mathcal{F}_M(A)$ for any $A \in \mathcal{A}_k$.*

Here we should explain about (non-commutative) complete local $k$-algebras.

**Definition 2.3** A $k$-algebra $A$ is said to be a complete local $k$-algebra if the following conditions hold:

(1) $A$ is a local $k$-algebra with unique maximal ideal $\mathfrak{m}_A$,

(2) the natural injection $k \to A$ induces an isomorphism $k \cong A/\mathfrak{m}_A$,

(3) $\mathfrak{m}_A/\mathfrak{m}_A^2$ is of finite dimension as a $k$-vector space, and

(4) the natural surjections $A \to A/\mathfrak{m}_A^n$ induce an isomorphism $A \cong \varprojlim A/\mathfrak{m}_A^n$.

EXAMPLE 2.4 Let $S = k\langle t_1, \dots, t_r\rangle$ be a non-commutative polynomial ring (or the tensor algebra). Then the non-commutative formal power series ring $T$ is defined as follows:

$$T = \varprojlim S/(t_1, \dots, t_r)^n,$$

which we denote by $k\langle\langle t_1, \dots, t_r\rangle\rangle$. It is easy to see that $T$ is a complete local $k$-algebra.

The following is easily seen only from the definition.

**Proposition 2.5** *$A$ is a complete local $k$-algebra if and only if it is of the form $T/I$ where $T = k\langle\langle t_1, \ldots, t_r\rangle\rangle$ and $I$ is an ideal of $T$ that is closed in $\mathfrak{m}_T$-adic topology, i.e. $I = \overline{I} := \bigcap_n (I + \mathfrak{m}_T^n)$.*

REMARK 2.6 (1) Any artinian local algebra $A \in \mathcal{A}_k$ is a complete local $k$-algebra.

(2) Complete local $k$-algebras are not necessarily noetherian. An ideal of a complete local $k$-algebra may not be closed.

As one of the typical examples, we consider $T = k\langle\langle x, y\rangle\rangle \supseteq (x) =$ (the two-sided ideal generated $x$). Since any element of $(x)$ is a finite sum of elements of the form $axb$ $(a, b \in T)$, we see that

$$\sum_{i=1}^{\infty} y^i x y^i \in \overline{(x)} \setminus (x).$$

In particular, $(x)$ is not a closed ideal of $T$, and hence $T/(x)$ is not a complete local $k$-algebra. Notice however that $T/\overline{(x)} \simeq k\langle\langle y\rangle\rangle = k[[y]]$.

REMARK 2.7 We can easily observe that every finitely generated left ideal of $T = k\langle\langle t_1, \ldots, t_r\rangle\rangle$ is free as a left $T$-module. For example, $\mathfrak{m}_T \cong T^{\oplus r}$ as left $T$-modules.

We can also prove the uniqueness of the parameter algebras and the universal liftings.

**Theorem 2.8** *Under the assumption of Theorem 2.2, the parameter algebra $P_0$ is unique up to $k$-algebra isomorphisms. Fixing the parameter algebra $P_0$, the universal lifting $\mathbb{L}_0$ is unique up to chain isomorphisms over $R\widehat{\otimes}_k P_0^{op}$.*

*Furthermore, if $r = \dim \mathrm{Ext}_R(M, M) (< \infty)$, then the parameter algebra $P_0$ is of the form $T/I$ where $T = k\langle\langle t_1, \ldots, t_r\rangle\rangle$ and $I$ is a closed ideal $T$ contained in $\mathfrak{m}_T^2$.*

We should note that every complete local $k$-algebra can be a parameter algebra. In fact, we can prove the following.

**Theorem 2.9** *Let $P$ be an arbitrary complete local $k$-algebra, and take a free resolution $\mathbb{F} = (F, d)$ of the left $P$-module $k = P/\mathfrak{m}_P$. Then there is a universal lifting of $\mathbb{F}$ of the form*

$$\mathbb{L} = (F\widehat{\otimes}_k P, \Delta).$$

*In particular, $P$ itself is the parameter algebra of the universal lifting $\mathbb{L}$.*

# 3 Some properties of parameter algebras

As stated in the previous section, a parameter algebra has the form $T/I$ where $T$ is a non-commutative formal power series ring and $I$ is a closed ideal of $T$. Some of the properties of $I$ can be reduced to the structural properties of the Yoneda algebra $\mathrm{Ext}_R^\bullet(\mathbb{F},\mathbb{F})$.

In the rest of the article, $R$ is a $k$-algebra and $\mathbb{F}=(F,d)$ is a projective complex of $R$-modules, as before.

**Theorem 3.1** *Suppose the following two conditions hold:*

(1) $r := \dim_k \mathrm{Ext}_R^1(\mathbb{F},\mathbb{F}) < \infty$

(2) $\ell := \dim_k \mathrm{Ext}_R^2(\mathbb{F},\mathbb{F}) < \infty$

*Let $P_0$ be the parameter algebra of the universal lifting of $\mathbb{F}$. Then we can find $\ell$ elements in the non-commutative formal power series ring with $r$ variables*

$$f_1,\dots,f_\ell \in T = k\langle\langle t_1,\dots t_r\rangle\rangle,$$

*so that $P_0$ is described as*

$$P_0 = T/\overline{(f_1,\dots,f_\ell)}.$$

**Corollary 3.2** *Under the assumption of the theorem, assume that* $\dim_k \mathrm{Ext}_R^2(\mathbb{F},\mathbb{F}) = 0$. *Then $P_0$ is a non-commutative formal power series ring.*

REMARK 3.3 Let $P_0$ be the parameter algebra of the universal lifting of $\mathbb{F}$. Take its residue ring by the closure of the commutator ideal, and we obtain a commutative $k$-algebra

$$Q_0 = P_0/\overline{[P_0,P_0]}.$$

Then $Q_0$ is the commutative parameter algebra for $\mathbb{F}$ in the sense of Theorem 1.1. In fact, if we restrict the functor $\mathcal{F}$ over $\mathcal{A}_k$ to its full subcategory $\mathcal{C}_k$, then the commutative algebra $Q_0$ pro-represents the restricted functor $\mathcal{F}|_{\mathcal{C}_k}$.

**Theorem 3.4** *Let $P_0$ be the parameter algebra of the universal lifting of $\mathbb{F}$. And assume $P_0$ is described as $P_0 = T/I_0$, where $T$ is a non-commutative formal power series ring and $I_0 \subseteq \mathfrak{m}_T^2$ is a closed ideal of $T$. Then there is an isomorphism of $k$-vector spaces:*

$$\mathrm{Ext}_R^1(\mathbb{F},\mathbb{F})^2 \cong \mathrm{Hom}_k(I_0/I_0\cap\mathfrak{m}_T^3, k),$$

*where the left hand side of the isomorphism means the subspace of $\mathrm{Ext}_R^2(\mathbb{F},\mathbb{F})$ generated by all the products of two elements in $\mathrm{Ext}_R^1(\mathbb{F},\mathbb{F})$.*

**Corollary 3.5** *Under the circumstances of the theorem, the following conditions are equivalent.*

$$I_0 \subseteq \mathfrak{m}_T^3 \quad\Leftrightarrow\quad \mathrm{Ext}_R^1(\mathbb{F},\mathbb{F})^2 = 0$$

In the rest, we assume that the projective complex $\mathbb{F} = (F, d)$ of $R$-modules is right bounded. And assume that $\mathbb{F}$ has the universal lifting $\mathbb{L}_0 = (F\widehat{\otimes}_k P_0, \Delta_0)$ with the parameter algebra $P_0$.

For any natural number $n$, we have a projective complex over $R \otimes_k (P_0/\mathfrak{m}_{P_0}^n)^{op}$;

$$\mathbb{L}_0^{(n)} = (F \otimes_k P_0/\mathfrak{m}_{P_0}^n,\ \Delta_0 \otimes_{P_0} P_0/\mathfrak{m}_{P_0}^n).$$

Since this complex is flat as a right $P_0/\mathfrak{m}_{P_0}^n$-module, it induces the functor between the derived categories;

$$\mathbb{L}^{(n)} \otimes_{P_0/\mathfrak{m}_{P_0}^n} - \ :\ D_+(P_0/\mathfrak{m}_{P_0}^n) \to D_+(R).$$

Notice that $k$ maps to $\mathbb{F}$ under this functor. And we have an algebra map between Yoneda algebras;

$$\mathrm{Ext}^\bullet_{P_0/\mathfrak{m}_{P_0}^n}(k, k) \to \mathrm{Ext}^\bullet_R(\mathbb{F}, \mathbb{F}).$$

Taking the inductive limit of such maps, we finally have the map

$$\rho^\bullet\ :\ \varinjlim \mathrm{Ext}^\bullet_{P_0/\mathfrak{m}_{P_0}^n}(k, k) \to \mathrm{Ext}^\bullet_R(\mathbb{F}, \mathbb{F}).$$

Note that this is also an algebra map. We are interested in how the map $\rho^i$ behaves for $i = 0, 1, 2, \ldots$.

(1) $\rho^0\ :\ \varinjlim \mathrm{Hom}_{P_0/\mathfrak{m}_{P_0}^n}(k, k) = k \to \mathrm{End}_R(\mathbb{F})$ is a natural embedding, hence it is always injective.

(2) $\rho^1\ :\ \varinjlim \mathrm{Ext}^1_{P_0/\mathfrak{m}_{P_0}^n}(k, k) = (\mathfrak{m}_{P_0}/\mathfrak{m}_{P_0}^2)^* \to \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$ is always bijective.

The following theorem is our final goal which states that $\rho^2$ behaves well as a comparison map of cohomology modules.

**Theorem 3.6** $\rho^2\ :\ \varinjlim \mathrm{Ext}^2_{P_0/\mathfrak{m}_{P_0}^n}(k, k) \to \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$ *is always injective.*

We should note that $\rho^2$ is not necessarily bijective.

# References


[1] B. Fantechi et al, *Fundamental Algebraic Geometry, Grothendieck's FGA explained*, Mathematical Survey and Monographs, vol. 123, Amer. Math. Soc. (2005).

[2] Robin Hartshorne, *Lectures on deformation theory*, available from http://math.berkeley.edu /~robin/.

[3] Michael Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. vol. 130 (1968), 208–222.

[4] Yuji Yoshino, *Universal lifts of chain complexes over non-commutative parameter algebras*, preprint (2007).