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RIEMANN–ROCH THEOREMS FOR VIRTUALLY SMOOTH SCHEMES

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ABSTRACT. We will present some recent results on an extension to virtually smooth schemes of the theorems of Grothendieck–Riemann-Roch and Hirzebruch–Riemann-Roch; we also define the virtual χ_{-y}-genus of a proper virtually smooth scheme, and show its polynomiality. These results were obtained jointly with L. Göttsche: complete proofs can be found in [FG].

CONTENTS

1. Introduction

The Riemann-Roch theorem for curves, in its original form, states that for a line bundle L on a smooth projective curve C one has

$$\dim H^0(X,L) - \dim H^0(X, K_X \otimes L^{-1}) = \deg L + 1 - g.$$

It is easy to generalize it to vector bundles, and indeed to any coherent sheaf. To extend it to higher dimensional smooth manifolds, one has to rewrite the left hand side as $\chi(X,L) = \sum(-1)^i \dim H^i(X,L)$; the theorem determines the Euler characteristic of a vector bundle E (or any element of $K^0(X)$, see below) on a smooth proper variety X in terms of intersection numbers of the Chern classes of E and of T_X, and it is called Hirzebruch-Riemann-Roch (HRR) Theorem.

Grothendieck applied his philosophy of studying morphisms instead of objects, and recast the theorem in a much more general form, producing a modified version of the Chern character, the τ class, which commutes with proper pushforward: this is called Grothendieck–Riemann–Roch Theorem (GRR). HRR is the special case of GRR where the target of the morphism is a point.

HRR allows one to define a number of interesting invariants of smooth proper varieties, such as the χ_{-y} and elliptic genus, that carry information about the topology and are invariant not only under isomorphism but also under smooth deformations. Indeed, these invariants go back to Hirzebruch's original proof of HRR.

Versions of GRR, and hence of HRR, have been developed for singular schemes (see [Fu] for a very readable account, including lots of history) and for DM algebraic stacks (by Toen in [T]).

In this paper we extend this circle of ideas (GRR, HRR and the χ_{-y} and elliptic genus) to the case of virtually smooth schemes, that is schemes admitting a (1-perfect) obstruction theory, which arise naturally in many defintions of enumerative invariants. We further assume these schemes to admit a global embedding in a smooth scheme (e.g., this is true for quasiprojective schemes).

The motivation for this paper are two-fold. On the one hand, there is an abstract interest in extending to virtually smooth schemes as many as possible of the classical constructions available for smooth varieties, in particular those that yield deformation invariant numerical results.

The more practical motivation is to provide some necessary tools for an ongoing joint project of L. Göttsche, T. Mochizuki, H. Nakajima and K. Yoshioka [GMNY] on K-theoretic Donaldson invariants and their generalizations.
In [GNY] K-theoretic Donaldson invariants have been introduced as holomorphic Euler characteristic of certain "determinant" line bundles on moduli spaces of stable sheaves on surfaces; in the rank two case their wallcrossing formulas have been determined via the Nekrasov partition function, under assumptions which eventually ensure that the

moduli spaces will be nonsingular.
Combining virtual Riemann Roch with results of Mochizuki [Mo] allows to extend these results to arbitrary rank, with no restrictions on the singularities of the moduli space.

The work presented here has two natural directions of development: one is to extend it to at least some virtually smooth DM stacks (see the section of comments, at the end of the paper); the other is to generalize further GRR by allowing also Y, and not just X, to be virtually smooth. The latter is very natural but at the moment we don't know of any applications.

The main difference between this exposition and that in [FG] is that here we try to be accessible to a wider audience, and in particular make it more evident where in the proof each assumption is used; in the comment section, we will discuss which of the assumptions are necessary even to be able to state the relevant results, and which are introduced only to make the proof work.

We have tried to give references for all definitions and results going beyond Hartshorne's text [Ha]; for the reader's convenience, we have tried whenever possible to refer to the book of Fulton [Fu] instead of the original literature.

I would like to thank the organizers of the Kinosaki Symposium for inviting me to present my work, and in particular Professor Ito for her generous hospitality and help.

2. Classical Riemann–Roch Theorems

2.1. **Grothendieck groups.** Let $K^0(X)$ be the Grothendieck group of locally free sheaves on a scheme X, i.e. the free abelian group generated by locally free sheaves modulo the relations $[E] = [E'] + [E/E']$ for every bundle E and every subbundle E'. The abelian group $K^0(X)$ is a ring, with $\oplus$ and $\otimes$ as operations, and is contravariant under arbitrary morphisms.

One can similarly define the Grothendieck group of coherent sheaves, $K_0(X)$; it is a $K^0(X)$ module, and there is a natural injective morphism of $K^0(X)$-modules $K^0(X) \to K_0(X)$ (since every locally free sheaf is coherent), which is an isomorphism iff X is smooth (see e.g. [Fu], Appendix B.8.3). $K_0(X)$ is covariant for proper morphism, with pushforward defined by $f_*[\mathcal{F}] = \sum(-1)^n R^n f_* \mathcal{F}$.

A special class of morphisms are the so-called perfect morphisms, i.e. those that factor into a closed embedding followed by a smooth map. For these one can define proper pushforward on K^0 and pullback on K_0 (see [Fu] Example 15.1.8 and references therein). The zero section of a vector bundle and any morphism with smooth target are examples of perfect morphisms: we will use both these examples.

2.2. **Perfect complexes.** We denote by $D_{coh}(X)$ the derived category of coherent sheaves on X; we denote by $D^{\leq 0}_{coh}(X)$ the subcategory of complexes whose positive degree cohomology vanishes, and by $D^b_{coh}(X)$ that of complexes with finitely many nonzero cohomology sheaves . We say that a complex $E \in D_{coh}(X)$ is perfect if it is locally isomorphic to a finite complex of locally free sheaves; a (global) resolution is an isomorphism of E with a finite complex of locally free sheaves. We say that a complex E is perfect of amplitude contained in $[a,b]$ if locally one can find resolutions of the form $[E^a \to E^{a+1} \to \ldots \to E^b]$, with each E^i locally free. For brevity, we also say that E is a $[a,b]$-perfect complex.

If $E \in D^b_{coh}(X)$ is a complex, then one can associate to it the class $[E] = \sum_i h^i(E) \in K_0(X)$; this is clearly invariant under quasiisomorphisms. If E is perfect and has a global resolution, then $[E] \in K^0(X)$. Its (locally constant) rank can then be defined as $\sum_i (-1)^i \operatorname{rk} E^i$, if $[E^a \to E^{a+1} \to \ldots \to E^b]$ is a resolution of E.

It is easy to see that a resolution always exists if X has enough locally frees, i.e., if every coherent sheaf is a quotient of a locally free sheaf. In particular, every embeddable scheme (i.e., one that can be realized as closed subscheme of a smooth scheme) has enough locally frees (see e.g. [Ha], Exercise III.6.8).

2.3. **Chern character and Todd class.** For a scheme X, we let $A^*(X)$ be the Chow ring with rational coefficient, and $A_*(X)$ the Chow group with rational coefficients. $A^*(X)$ is a graded ring, contravariant under arbitrary morphisms, and $A_*(X)$ is a graded $A^*(X)$ module, covariant under proper morphisms. They reproduce in an algebraic context the same properties of cohomology and homology in the usual topological settings.

Let E be a rank r vector bundle on a scheme X, x_i its Chern roots (see e.g., [Fu] Remark 3.2.3). The Chern character ch and the Todd genus td are defined as

$$\operatorname{ch}(E) := \sum_{i=0^r} e^{x_i} \qquad \operatorname{td}(E) := \prod_{i=0}^{r} \frac{x_i}{1-e^{-x_i}}.$$

Remark 2.1. There are unique functorial extensions

$$\operatorname{ch} : K^0(X) \to A^*(X) \qquad \text{ring hom}$$

$$\operatorname{td} : K^0(X) \to A^*(X)^\times \qquad \text{group hom}$$

where $A^*(X)^\times$ is the group of multiplicative units in $A^*(X)$. Here functorial means that they commute with pullback via arbitrary morphisms.

2.4. **Statement of the Theorems.** The theorem of Grothendieck Riemann Roch measures the failure of the Chern character to commute with proper pushforward between smooth varieties in terms of Todd classes of the tangent bundles involved.

Theorem 2.2. (Grothendieck–Riemmann–Roch) *Let $f : X \to Y$ be a proper morphism of smooth varieties, $V \in K^0(X)$ (e.g., V a vector bundle on X). Let $[X] \in A_{\dim X}(X)$ the fundamental class. Then:*

$$\mathrm{ch}(f_*(V)) \cdot \mathrm{td}(T_Y) \cap [Y] = f_*(\mathrm{ch}(V) \cdot \mathrm{td}(T_X) \cap [X]) \in A_*(Y).$$

Note that f is perfect, hence f_*V is well defined as an element in $K^0(Y)$; moreover, we can rewrite the statement as

$$\mathrm{ch}(f_*V) \cap [Y] = f_*(\mathrm{td}(T_f) \cdot \mathrm{ch}(V) \cap [X]).$$

GRR can be extended to singular schemes, see [Fu] Chapter 18, and also

Theorem 2.3. (Hirzebruch–Riemann–Roch) *Let X be a smooth proper variety, $V \in K^0(X)$. Then*

$$\chi(X, V) = \deg(\mathrm{ch}(V) \cdot \mathrm{td}(T_X) \cap [X]).$$

Remark 2.4. Hirzebruch Riemann Roch is Grothendieck Riemann Roch in the case where Y is one point.

3. Virtually smooth schemes

3.1. **Hidden smoothness.** If X is a fine moduli space, then for every point $x \in X$ deformation theory allows to construct an *obstruction space* $T^2_x X$, with the property that, locally analytically or formally near x, X can be described as the zero locus of $\dim T^2_x X$ functions whose differential vanishes at x in $\dim T_x X$ variables.

In particular, this implies that X is smooth at x if (but not only if!) $T^2_x X$ is zero; more generally, $\dim_x X \geq d(x) := \dim T_x X - \dim T^2_x X$. We say that X has *expected dimension* d if $d(x) = d$ at every $x \in X$.

Example 3.1. If X is a fine moduli scheme (or DM algebraic stack) of surfaces, $x = [S]$; then $T_x X = H^1(S, T_S)$ and $T^2_x X = H^2(S, T_S)$; since in this case necessarily $H^0(S, T_S) = 0$, one has that $d([S]) = -\chi(S, T_S)$, and by Riemann Roch it only depends on the Chern numbers of S.

Schemes of expected dimension d, usually arising as moduli schemes, may in fact have many components of varying dimensions and arbitrary singularities.

When Gromov Witten invariants were in the process of being defined, Kontsevich [K] proposed a *hidden smoothness* philosophy: namely, that one should be able to extend to schemes with expected dimension d several properties of d–dimensional smooth schemes, by introducing an additional, "hidden" structure that is smooth in a suitable sense.

3.2. **Virtually smooth schemes.** Let X be a scheme, $L_X \in D^{\leq 0}_{coh}(X)$ its cotangent complex. The definition is somewhat involved, but for the purpose of this paper it is enough to know that if $i : X \to M$ is a closed embedding in a smooth scheme,

$$\tau_{\geq -1} L_X = \left[\mathcal{I}_X/\mathcal{I}_X^2 \to i^*\Omega_M|_X\right].$$

In the following we write $\tilde{L}_X$ instead of $\tau_{\geq -1} L_X$.

Definition 3.2. An *obstruction theory* for a scheme X is a pair (E, ϕ) such that
(1) E is a $[-1, 0]$ perfect complex in $D^b_{coh}(X)$;
(2) $\phi : E \to \tilde{L}_X$ is a morphism with $h^0(\phi)$ iso, $h^{-1}(\phi)$ onto.

Remark 3.3. (a) In practice E is an obstruction theory iff $\forall x \in X$, one has natural isomorphisms $h^0(E^\vee(x)) = T_x X$ and $h^1(E^\vee(x)) = T^2_x X$.
(b) If $\psi : E \to L_X$ is a morphism, then it induces a natural morphism $\phi = \tau_{\geq -1}\psi : E = \tau_{\geq -1} E \to \tilde{L}_X$; so if we formulate Definition 3.2(2) with L_X instead of $\tilde{L}_X$ we get a stronger, if more natural, assumption.

Definition 3.4. A **virtually smooth scheme** X of dimension d is a scheme X together with an obstruction theory (E, ϕ) of rank d admitting a global resolution.

Remark 3.5. When we say "Let X be a virtually smooth scheme" we mean that we have *fixed* an obstruction theory $\phi : E \to L_X$. A scheme X may not have any obstruction theory, or it may have many of different dimensions.

3.3. **Examples.** Any smooth scheme X of dimension d has a natural structure of virtually smooth scheme of dimension d, with $E = L_X = \Omega_X$ in degree zero.

More generally, assume that X is a local complete intersection (lci) scheme, i.e., for one (or every) closed embedding $i : X \to M$ in a smooth variety, then $\mathcal{I}_X/\mathcal{I}_X^2$ is locally free. Then $E = L_X$ is a natural obstruction theory on X.

Therefore, virtually smooth schemes generalize both smooth and lci schemes.

The moduli scheme of morphisms from a projective curve C to a smooth projective variety V of homology class β is virtually smooth of dimension

$$d := (g - 1)\dim V + c_1(T_V) \cdot \beta.$$

The Hilbert scheme of closed subschemes of dimension ≤ 1 of a Calabi Yau threefold is virtually smooth of dimension zero.

The moduli scheme of stable sheaves on a smooth projective surface S is virtually smooth, with tangent space at a point $[E]$ being $\mathrm{Ext}^1(E, E)$ and obstruction space $\mathrm{Ext}^2(E, E)$.

These obstruction theories can be used to define Gromov-Witten, Donaldson-Thomas, and Donaldson invariants, respectively.

3.4. **The intrinsic normal cone.** Let X be a scheme, E an obstruction theory of dimension d, and $[E^{-1} \to E^0]$ a global resolution of E. One can naturally associate to these data a cone $C \subset E_1 := \operatorname{Spec}\operatorname{Sym} E^{-1}$, of pure dimension equal to $d + \operatorname{rk} E^{-1} = \operatorname{rk} E^0$, and invariant under the natural action of $E_0 := \operatorname{Spec}\operatorname{Sym} E^0$ via fiberwise translation.

Indeed, there is a natural cone stack $\mathfrak{C}_X$ over X, of pure dimension zero, which can be defined as the stack quotient $[C_{X/M}/T_M|_X]$ for any closed embedding of X in a smooth scheme M; the fact that E is an obstruction theory translates precisely into having a closed embedding of $\mathfrak{C}_X$ inside $\mathfrak{E} := [E_1/E_0]$, independent of the choice of resolution, and $C \subset E_1$ is just the inverse image of $\mathfrak{C}_X$.

Proposition 3.6. *Let $p : C \to X$ be the natural projection. Then*

$$\tau_C(\mathcal{O}_C) = p^*(\operatorname{td} E_0) \cap [C].$$

This result is Proposition 3.1 in [FG]; it is proven by reducing it to the a similar statement, where C is replaced by $C_{X/M}$, the normal cone of some closed embedding of X into a smooth variety M, and E_0 is replaced by $T_M|_X$. If one assumes that the τ map could be extended to Artin stacks with the same properties, then the Proposition takes the simple form $\tau_{\mathfrak{C}}(\mathcal{O}_{\mathfrak{C}}) = [\mathfrak{C}]$.

3.5. **Known results.**

Definition 3.7. Let X be a virtually smooth scheme of dimension d. Then one can define for X the following:

(1) a *virtual fundamental class* $[X]^{vir} \in A_d(X)$;
(2) a *virtual structure sheaf* $\mathcal{O}_X^{vir} \in K_0(X)$;
(3) a *virtual tangent bundle* $T_X^{vir} \in K^0(X)$.

We give an explicit definition, based on the choice of a resolution $[E^{-1} \to E^0]$ of E. Let $C \subset E_1$ be the cone introduced in §3.4; recall that C has pure dimension $\operatorname{rk} E^0$. Let $s_0 : X \to E_1$ be the zero section; since E_1 is a vector bundle, s_0 is a perfect morphisms.

(1) Define $[X]^{vir} := s_0^*[C]$, where $s_0 : X \to E_1$ is the zero section.
(2) Define $\mathcal{O}_X^{vir} := s_0^*(\mathcal{O}_C) := \sum_i \operatorname{Tor}_i^{E_1}(\mathcal{O}_C, \mathcal{O}_{s_0(X)})$.
(3) Using §2.2, we define $T_X^{vir} := [E^\vee] \in K^0(X)$.

The fact that T_X^{vir} does not depend on the resolution chosen follows from the arguments in §2.2.

Theorem 3.8. *$[X]^{vir}$ and $\mathcal{O}_X^{vir}$ only depend on the obstruction theory up to isomorphism in $D^b_{coh}(X)$.*

The independence on the choice of resolution is important because usually even when the obstruction theory has a resolution there is no natural one. To prove the Theorem, one can compare two different obstruction theories with a third, dominating both in a natural way (as done in [BF]).

Alternatively, one can show that $[X]^{vir}$ and $\mathcal{O}_X^{vir}$ can be defined without using the global resolution, by replacing C by $\mathfrak{C}_X$, E_1 by $\mathfrak{E}$, and using intersection theory on Artin stacks as in [Kr].

Remark 3.9. If X is a smooth scheme with the natural virtual smooth structure, then we can choose $E^0 = \Omega_X$, E^{-1} the rank zero bundle; then $C = E_1 = X$, and $\mathcal{O}_X^{vir} = \mathcal{O}_X$, $[X]^{vir} = [X]$, $T_X^{vir} = [T_X] \in K^0(X)$.

3.6. **Families of virtually smooth schemes.** One of the key properties of the objects we define in this paper (virtual versions of Euler characteristic, χ_{-y} and elliptic genus) is that, like their classical counterparts, they are deformation invariant. Here we explain what we mean by this.

Let $f : X \to B$ be a morphism of schemes; one can define the relative cotangent complex $L_f \in D^{\leq 0}_{coh}(X)$ (or $L_{X/Y}$) by showing that there is a natural morphism $f^*L_B \to L_X$ and letting L_f be the mapping cone of this morphism.

If X is embeddable, we can explictly construct $\tau_{\geq -1}L_f$ as follows. Write f as $p \circ i$, where $i : X \to W$ is a closed embedding and $p : W \to B$ is a smooth morphism: for instance, if $j : X \to M$ is a closed embedding in a smooth variety, you can take $W = B \times M$, $i = (f, j)$ and $p = p_B$. Then $\tau_{\geq -1}L_f = [\mathcal{I}_X/\mathcal{I}_X^2 \to i^*\Omega_{W/B}]$, where I_X is the ideal sheaf of X in W.

Definition 3.10. Let $f : X \to B$ be a morphism, with B a smooth scheme. A *relative obstruction theory* of (relative) dimension d for f (or for X over B) is a morphism $\phi : E \to L_f \in D^{\leq 0}_{coh}(X)$ such that E is a rank d perfect complex in $D^{[-1,0]}_{coh}(X)$, $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. In particular, for every $b \in B$, the pullback of E_b of E to X_b is an obstruction theory for X_b.
A *family of virtually smooth schemes* with base B is a morphism $f : X \to B$ together with a relative obstruction theory of dimension d for f; this gives each fiber X_b over a closed point b a natural structure of virtual smooth scheme of dimension d. The family is proper, or a family of proper virtually smooth schemes, if the morphism f is proper.

Note that in the definition of virtually smooth family no flatness condition is imposed: in particular, the fibers of f may have different dimensions.

All the deformation invariance results described here follow immediately from the following Lemma, which is Lemma 3.15 in [FG]. Its proof is based on the principle of conservation of number (as in [Fu], Chapter 10) together with the definition and properties of a relevant version of the virtual fundamental class. It is usually more useful to apply this lemma then to know its proof.

Lemma 3.11. *Let $f : X \to B$ be a proper family of virtually smooth scheme, with basis a smooth, connected variety B; let $\alpha \in A^*(X)$. Then*

$$\deg(\alpha \cap [X_b]^{vir})$$

does not depend on the choice of b, closed point in B.

4. Virtual RR theorems

4.1. Statements.

Theorem 4.1. (virtual Grothendieck–Riemann–Roch) *Let X be a virtually smooth scheme, $V \in K^0(X)$, Y a smooth scheme, $f : X \to Y$ a proper morphism. Then the following equality holds in $A_*(Y) \otimes \mathbb{Q}$:*

$$\mathrm{ch}(f_*(V \otimes \mathcal{O}_X^{vir})) \cdot \mathrm{td}(T_Y) \cap [Y] = f_*(\mathrm{ch}(V) \cdot \mathrm{td}(T_X^{vir}) \cap [X]^{vir}).$$

Corollary 4.2. (virtual Hirzebruch–Riemann-Roch theorem) *If X is a proper virtually smooth scheme and $V \in K^0(X)$, then*

$$\chi^{vir}(V) := \chi(V \otimes \mathcal{O}_X^{vir}) = \deg(\mathrm{ch}(V) \cdot \mathrm{td}(T_X^{vir}) \cap [X]^{vir}).$$

Remark 4.3. (1) HRR is the special case of GRR when Y a point; the applications use HRR and deformation invariance.
(2) The differences between the two statement of GRR can be summarized as follows:

- replace X smooth by X virtually smooth in the assumptions;
- replace $V = V \otimes \mathcal{O}_X$ by $V \otimes \mathcal{O}_X^{vir}$;
- replace T_X by T_X^{vir};
- replace $[X]$ by $[X]^{vir}$.

4.2. **The τ class.** A key point of the proof is the so-called τ class. For any embeddable scheme X, there is a group homomorphism $\tau_X : K_0(X) \to A_*(X)$, enjoying the following properties:

1. *covariance*: for every proper morphism $f : X \to Y$ of embeddable schemes, $f_* \circ \tau_X = \tau_Y \circ f_*$.
2. *smooth scheme*: if X is smooth and $V \in K^0(X) = K_0(X)$, then $\tau_X(V) = (\mathrm{ch}(V) \cdot \mathrm{td}(T_X)) \cap [X]$;
3. K^0*-module*: for any $F \in K_0(X)$ and $V \in K^0(X)$, one has $\tau_X(V \otimes F) = \mathrm{ch}(V) \cap \tau_X(F)$;
4. *lci contravariance*: if $f : X \to Y$ is an lci (hence perfect by ?) morphism of embeddable schemes, then for every $F \in K_0(X)$ one has $f^*(\tau_Y(F)) = \mathrm{td}(T_f) \cap \tau_X(f^*F)$.

Note that by the smooth case and the K^0-module properties, if X is smooth and $V \in K^0(X) = K_0(X)$, then $\tau_X(V) = (\mathrm{ch}(V) \cdot \mathrm{td}(T_X)) \cap [X]$.

The class τ_X plays a fundamental role in the generalization of the RR theorems to singular schemes, in that the classical GRR can be restated as the covariance property for τ; therefore, extending to singular schemes the homomorphism τ with its property yields the natural generalization of GRR.

4.3. **Outline of the proof of virtual GRR.** The proof is based on a series of reduction steps.

Step 1. It is enough to show that $\tau_X(\mathcal{O}_X)^{vir} = \mathrm{td}(T_X^{vir}) \cap [X]^{vir}$; this reduction is an easy consequence of the covariance, module and smooth case properties of τ. Notice that this way we do not have any more $V \in K^0(X)$ in the statement; moreover, notice the analogy with the smooth scheme property of the τ class. Indeed, one could view this as a as a virtual analogue of the smooth scheme property.

Step 2. It is enough to show that for *one* resolution $[E^{-1} \to E^0]$ of E one has

$$s_0^*(\tau_{E_1}(\mathcal{O}_C)) = \mathrm{td}\, E_0 \cap [X]^{vir},$$

where $C \subset E_1$ is the inverse image of the intrinsic normal cone as explained in §3.4. This uses only the lci property of τ.

Step 3. Finally, one reduces (by covariance property of τ) to Prop. 3.6. This way, even the obstruction theory has been eliminated and we are reduced to a property of normal cones.

5. Virtual χ_{-y}–genus

5.1. **Classical χ_{-y}-genus.** Let E be a vector bundle on a scheme X. Define

$$\Lambda_t(E) := \sum [\Lambda^i E] t^i \in K^0(X)[t], \qquad S_t(E) := \sum [S^i E] t^i \in K^0(X)[[t]].$$

Define a group homomorphism $\Lambda_t : K^0(X) \to K^0(X)[[t]]^\times$ by

$$\Lambda_t([E] - [F]) = \Lambda_t(E) \cdot S_{-t}(F).$$

If X is a smooth proper scheme, the χ_{-y} genus is defined as

$$\chi_{-y}(X) := \chi(\Lambda_{-y}(\Omega_X)) = \sum_{n \geq 0} (-y)^n \chi(\Omega_X^n) \in \mathbb{Z}[y].$$

This notation is standard but can be confusing for the non-expert: indeed in χ_{-y} and in Λ_t the index should be viewed as a variable in a polynomial (with coefficients in $K^0(X)$ and $\mathbb{Z}$, respectively).

The χ_{-y}-genus is a polynomial of degree less than or equal to $\dim X$; since the topological Euler characteristic $e(X)$ is equal to

$$\sum_{p,q} \dim H^q(X, \Omega_X^p) = \sum_p \chi(\Omega_X^p),$$

one has $e(X) = \chi_{-1}(X)$. Analogously, the signature $\sigma(X)$ equals $\chi_1(X)$.

5.2. **Definition and polynomiality.** Let X be a proper virtually smooth scheme of dim d. Define $\Omega_X^{n,vir}$ by

$$\Lambda_{-y}((T_X^{vir})^\vee) = \sum_{n \geq 0} (-y)^n \Omega_X^{n,vir} \in K^0(X)[[y]].$$

Note that $\Omega^{n,vir} \in K^0(X)$ need not be zero for $n > d$, the virtual dimension of X.

Then we define the *virtual χ_{-y}-genus* of X as

$$\chi_{-y}^{vir}(X) := \chi^{vir}(\Lambda_{-y}((T_X^{vir})^\vee)) = \sum_{n\geq 0}(-y)^n \chi^{vir}(\Omega_X^{n,vir}) \in \mathbb{Z}[[y]].$$

Theorem 5.1. (1) *$\chi_{-y}^{vir}(X)$ is a polynomial of degree $\leq d$;*
(2) *$\chi_{-1}^{vir}(X) = \deg(c_d(T_X^{vir}) \cap [X]^{vir})$;*
(3) *if d is odd, then $\chi_1^{vir}(X) = 0$.*

The proof of this (Theorem [FG], Theorem 4.5) only depends on the statement of HRR and some standard techniques in manipulating formal power series with coefficients in a graded ring. We do not include a proof since we have given two in [FG]: one (presented directly after the statement) is very short, the other (in the Appendix) is very detailed and suitable also for non-experts.

We can therefore define the virtual Euler char $e^{vir}(X) := \chi_{-1}^{vir}(X)$ and the virtual signature $\sigma^{vir}(X) := \chi_1^{vir}(X)$; they are deformation invariant and agree with the classical definition if X is smooth.

More generally, for X a proper, virtually smooth scheme of dim d and $V \in K^0(X)$, we define

$$\chi_{-y}^{vir}(X, V) := \chi^{vir}(V \otimes \Lambda_{-y}((T_X^{vir})^\vee)) \in \mathbb{Z}[[y]].$$

Theorem 5.2. (1) *$\chi_{-y}^{vir}(X, V)$ is a polynomial of degree $\leq d$;*
(2) *$\chi^{vir}(V \otimes \Omega_X^{n,vir}) = 0$ if $n > d$.*

The proof is very similar to that of the previous theorem. Again, $\chi_{-y}^{vir}(X, V)$ is deformation invariant and agrees with the classical definition if X is smooth.

Theorems 5.1 and 5.2 are an immediate consequence of deformation invariance in case X is virtually smoothable: i.e., it is a fiber of a family of vortually smooth schemes with smooth, connected bases B whose general fiber is smooth with obstruction theory equal to the cotangent bundle.

6. Other results and comments

6.1. **Other results.** In the paper [FG], some more results are extended from smooth to virtually smooth proper schemes. In particular,

(1) We prove a weak virtual version of Serre's duality theorem, namely if X is a proper virtually smooth scheme of dimension d, then for every $V \in K^0(X)$

$$\chi^{vir}(V) = (-1)^d \chi^{vir}(V^\vee \otimes \det(T_X^{vir})^\vee).$$

(2) We define a virtual analogue of the elliptic genus, and prove suitable weak modularity properties, generalizing the classical ones.

(3) We prove virtual localization formulas for χ^{vir}, χ^{vir}_{-y} and the virtual elliptic genus, in the same set-up as Graber–Pandharipande's virtual localization formula for $[X]^{vir}$.

6.2. **Relation with dg-schemes.** Ciocan-Fontanine and Kapranov have proven in [CF-K3] virtual GRR and HRR under the additional assumption that X is a quasiprojective scheme and that its obstruction theory comes from a structure of $[0,1]$–dg scheme.

In general, constructing a dg-scheme structure (see [CF-K1], [CF-K2]) is much more difficult than an obstruction theory. It is not known whether every obstruction theory is induced by a $[0,1]$-dg scheme structure.

The applications in [GMNY] require to apply the virtual RR theorem to a DM moduli stack $\mathcal{M}$ of stable sheaves with fixed determinant. This can be done easily, since the coarse moduli space M of $\mathcal{M}$ is a quasiprojective scheme (hence embeddable), and the natural obstruction theory on $\mathcal{M}$ descends to M for trivial reasons, so one can easily reduce the problem to working with M and our theorem applies.

It is expected (and indeed claimed in [CF-K1]) that the moduli stack $\mathcal{M}$ carries a dg structure induced by the one on dg-Quot, and it is possible that such a structure descends to M. However, this is for the moment not explicitly proven or claimed anywhere; in fact, a theory of dg-DM stacks has not been explicitely developed yet, to the best of our knowledge.

6.3. **Possible developments.** The theorem of virtual GRR as stated has an asymmetric formulation, in that X is supposed to be virtually smooth, while Y is smooth. A more natural statement would be obtained as follows.

First, define a morphism of virtually smooth schemes $f:(X,E_X)\to(Y,E_Y)$ to be a morphism $f:X\to Y$ together with a morphism $f^\sharp : f^*E_Y\to E_X$ in $D^b(X)$ such that the composition of $f^\sharp$ with the structure map $E_X\to\tilde{L}_X$ is the same as the composition of the structure morphism $f^*E_Y\to f^*\tilde{L}_Y$ with the natural morphism $f^*\tilde{L}_Y\to\tilde{L}_X$, and moreover such that the mapping cone E_f of $f^\sharp$ is $[-1,0]$ perfect.

Then, define a virtual τ homomorphism for virtually smooth schemes, $\tau_X^{vir}:K^0(X)\to A^*(X)$ by

$$\tau_X^{vir}(V):=\mathrm{ch}(V)\cdot\mathrm{td}(T_X^{vir})\cap[X]^{vir}.$$

Also define, for every proper morphism $f:X\to Y$ of virtually smooth schemes, a virtual pushforward morphism $f_*^{vir}:K^0(X)\to K^0(Y)$, and for every morphism of virtually smooth schemes as virtual pullback $f^*_{vir}:A_*(X)\to A_*(Y)$, in such a way that the usual properties of τ hold for τ^{vir} if we replace all the morphisms by their virtual versions (note that $f_*:A_*(X)\to A_*(Y)$ and $f^*:K^0(Y)\to K^0(X)$ coincide with their virtual counterpart, since their definition in the classical case doesn't use smoothness.

This would in particular imply virtual GRR, and really provide an extension of the τ classes to virtually smooth schemes with properties analogous to those for smooth schemes. There is currently work in progress in this direction by Göttsche and myself and by C. Manolache.

It would also be natural to do the same replacing everywhere embeddable schemes with quasiprojective Deligne-Mumford stacks, and correspondingly replacing the results of Fulton on the τ class for schemes by those of [T] for algebraic stacks.

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