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NONRATIONAL WEIGHTED HYPERSURFACES

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1. Introduction

We say that a normal projective variety defined over the field $\mathbb{C}$ of complex numbers is a $\mathbb{Q}$-*Fano* variety if its anticanonical divisor is an ample $\mathbb{Q}$-Cartier divisor. In [7], Kollár gives examples of nonrational Fano manifolds of arbitrary dimension ≥ 3.

Theorem 1.1 ([7], Theorem 4)**.** *If $d \geq 2\lceil (n+3)/3 \rceil$ then a very general hypersurfaces $X_d \subset \mathbb{P}^{n+1}$ of degree d defined over $\mathbb{C}$ is not ruled.*

By convention we say that X_d is *very general* when it does not belong to countable union of suitable proper closed subvarieties. In this article, we apply Kollár's techniques to weighted hypersurfaces. We show that, under certain conditions on the weights and the degree, the weighted hypersurface is not ruled, in particular, it is nonrational (Theorem 1.7). As a result, we obtain nonrational $\mathbb{Q}$-Fano threefolds with at most terminal singularities and infinitely many families of nonrational $\mathbb{Q}$-Fano varieties with at most log terminal singularities of arbitrary dimension ≥ 4 (cf. Section 3). These examples are rationally connected because a normal projective variety, defined over $\mathbb{C}$, with at most log terminal singularities whose anticanonical divisor is nef and big is rationally connected ([11]). Let us begin with basic definitions.

Definition 1.2. Let X be a variety of dimension n over a field k.

- We say that X is *rational* if there is a birational map $\mathbb{P}^n_k \dashrightarrow X$.
- We say that X is *ruled* (resp. *uniruled*) if there is a variety Y of dimension $n-1$ over k and a birational map (resp. dominant rational map) $Y \times \mathbb{P}^1_k \dashrightarrow X$.
- In positive characteristics, we say that X is *separably uniruled* if the above rational map $Y \times \mathbb{P}^1_k \dashrightarrow X$ is also separable.
- Let $\bar{k}$ be an algebraic closure of k. We say that X is *geometrically ruled* if $X_{\bar{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is ruled.

Definition-Lemma 1.3. Let $c_0, \ldots, c_n$ be positive integers and k a field. The *weighted projective space* $\mathbb{P}_k(c_0, \ldots, c_n)$ over k is defined by

$$\mathbb{P}_k(c_0, \ldots, c_n) := \operatorname{Proj} k[x_0, \ldots, x_n],$$

where $k[x_0, \ldots, x_n]$ is the graded polynomial ring with $\deg x_i = c_i$. The variables $x_0, \ldots, x_n$ are called *homogeneous coordinates*.

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One dimensional torus $\mathbb{G}_{\mathtt{m}} = \operatorname{Spec} k[t, t^{-1}]$ acts on $\mathbb{A}_k^{n+1}$ by $x_i \mapsto x_i \otimes t^{c_i}$ and then $\mathbb{P}_k(c_0, \dots, c_n)$ is the geometric quotient $(\mathbb{A}_k^{n+1} \setminus \{0\})/\mathbb{G}_{\mathtt{m}}$.

Let X be a closed subscheme of the weighted projective space $\mathbb{P} = \mathbb{P}_k(c_0, \dots, c_n)$ and let $\tau \colon \mathbb{A}_k^{n+1} \to \mathbb{P}$ be the canonical morphism. The *punctured affine cone* C_X^* of X is defined by $C_X^* = \tau^{-1}(X)$ and the *affine cone* C_X of X is the scheme theoretic closure of C_X^* in $\mathbb{A}_k^{n+1}$.

Definition 1.4. A closed subscheme X in $\mathbb{P}_k(c_0, \dots, c_n)$ is called *quasi smooth* if its affine cone C_X is smooth outside the origin.

For details of a weighted projective space, we refer the reader to [4]. In this article, we mainly consider the weighted projective space $\mathbb{P}(a_0, \dots, a_n, b)$, where the homogeneous coordinates are $x_0, \dots, x_n$ and y with $\deg x_i = a_i$ and $\deg y = b$. For a field k and a positive integer d, we denote by $H_d(k)$ the k-vector space $k[x_0, \dots, x_n]_d$, the degree d part of the graded ring $k[x_0, \dots, x_n]$ whose grading is given by $\deg x_i = a_i$.

Before stating the main theorems, we need to introduce conditions on positive integers $p, n, a_0, \dots, a_n, b$ and d.

Condition 1.5.

(1) $n \geq 3$ and $a_0 = 1$.
(2) $d = pb + 1$.
(3) $\gcd\{a_1, \cdots, a_n\} = 1$ and there are at least two i among $1, \cdots, n$ such that a_i is coprime to p.
(4) $\sum_{i=0}^n a_i < d < \sum_{i=0}^n a_i + b$.
(5) For any algebraically closed field $\Bbbk$ of characteristic p, a general weighted hypersurfaces of degree d in $\mathbb{P}_{\Bbbk}(a_1, \dots, a_n)$ is quasi smooth.

Condition 1.6. (Omit).

We give up introducing Condition 1.6 above because it is technical. Now we can state the main theorems of this article.

Theorem 1.7. *Assume that* $(p, \{a_i\}, b, n, d)$ *satisfies* Condition 1.5 *and* 1.6. *Then, the weighted hypersurface*

$$X_f := (y^p x_0 - f(x_0, \dots, x_n) = 0) \subset \mathbb{P}_{\mathbb{C}}(a_0, a_1, \dots, a_n, b)$$

of degree d *is a non-ruled log terminal* $\mathbb{Q}$*-Fano variety of dimension* n *for a very general* $f = f(x_0, \dots, x_n) \in H_d(\mathbb{C})$.

Theorem 1.8. *Assume that* $(p, \{a_i\}, b, n, d)$ *satisfies* Condition 1.5 *and* 1.6. *Let* $\Bbbk$ *be an algebraically closed field of characteristic* p. *Then, the weighted hypersurface*

$$X_f := (y^p x_0 - f(x_0, \dots, x_n) = 0) \subset \mathbb{P}_{\Bbbk}(a_0, a_1, \dots, a_n, b)$$

is not separably uniruled for a general $f = f(x_0, \dots, x_n) \in H_d(\Bbbk)$.

Suppose that $(p, \{a_i\}, b, n, d)$ satisfies Condition 1.5. Then, although it is not obvious, we see that the weighted hypersurface X_f defined over $\mathbb{C}$ is quasi smooth for a general $f \in H_d(\mathbb{C})$. In particular, it has only quotient singularities. The second inequality in Condition (1.5.4) implies that X_f is a $\mathbb{Q}$-Fano variety. The following result of Matsusaka reduces Theorem 1.7 to 1.8 (cf. [9, Section 4.4]).

Theorem 1.9 ([10],Appendix, Theorem 1.1, [8], IV, Theorem 1.6)**.** *Let R be an excellent discrete valuation ring and X a normal irreducible scheme. Let T be* $\operatorname{Spec} R$ *and $\varphi\colon X \to T$ a proper surjective morphism with connected fibers. Then the following assertions hold.*

(1) *If the generic fiber of φ is ruled over the quotient field of R, then every irreducible component of the special fiber of φ is ruled over the residue field of R.*
(2) *If the generic fiber of φ is geometrically ruled, then every reduced irreducible component of the special fiber of φ is geometrically ruled.*

The following Lemma, which is due to Kollár, is a key to the proof of Theorem 1.8.

Lemma 1.10 ([7], Lemma 7)**.** *Let X be a smooth proper variety and $\mathcal{M}$ a big line bundle on X. Assume that there is an injection $\mathcal{M} \hookrightarrow \Omega^i_X$ for some $i > 0$. Then X is not separably uniruled.*

A line bundle $\mathcal{L}$ on a normal projective variety is said to be *big* if some positive multiple of $\mathcal{L}$ defines a birational map onto its image.

2. Sketch of the proof of main theorems

As we explained in the previous section, Theorem 1.7 follows from Theorem 1.8. By Lemma 1.10, Theorem 1.8 is proved if we construct a desingularization $\varphi\colon Y \to X_f$ and a big line bundle on Y which is contained in Ω^i_Y for some i. Throughout this section, we assume the following.

Assumption 2.1.

- $(p, \{a_i\}, b, n, d)$ satisfies both Condition 1.5 and 1.6.
- We work over an algebraically closed field $\Bbbk$ of characteristic p.
- The weighted homogeneous polynomial $f = f(x_0, \ldots, x_n)$ is a general element of $H_d(\Bbbk)$ and $X = X_f$.
- We denote by A the integer

$$A = d - \sum_{i=0}^{n} a_i.$$

As a first step, we describe the singularities of X without proofs and construct a desingularization of X.

Lemma 2.2. *$X \cap \mathrm{D}_+(x_0)$ has only isolated singularities which are isomorphic to the origin of the hypersurface defined by one of the following equations.*

$$\nu^p = \begin{cases} \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{n-1}\xi_n + g_1 & \textit{if } n \textit{ is even },\\ \xi_1^2 + \xi_3\xi_4 + \cdots + \xi_{n-1}\xi_n + g_2 & \textit{if } n \textit{ is odd and } p \neq 2,\\ \xi_1^2 + \xi_2\xi_3 + \cdots + \xi_{n-1}\xi_n + \xi_1^3 + g_3 & \textit{if } n \textit{ is odd and } p = 2,\end{cases}$$

where $g_i = g_i(\xi_1, \ldots, \xi_n)$ consists of monomials of degree ≥ 3 and the coefficient of ξ_1^3 in g_3 is zero. Moreover, these singularities can be resolved by successive blow ups.

Put

$$X_{\mathrm{qs}} := X \setminus (\mathrm{Sing}(X) \cap \mathrm{D}_+(x_0)) \quad \text{and} \quad U_{\mathrm{qs}} := X_{\mathrm{qs}} \cap \mathrm{D}_+(x_0 \cdots x_n y).$$

The affine cone $C_{X_{\mathrm{qs}}}$ is smooth outside the origin, and this yields the following lemma.

Lemma 2.3. *$U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$ is a toroidal embedding without self-intersection.*

Here, we give a definition of a toroidal embedding. For details, we refer the reader to [6].

Definition 2.4. Let X be an algebraic variety defined over an algebraically closed field k and U a smooth Zariski open subset of X.

We say that $U \subset X$ is a *toroidal embedding* if for every closed point $x \in X$ there is a pair (S, s) of an affine toric variety S and its point s, and an isomorphism of k-local algebras $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{S,s}$ such that the ideal in $\hat{\mathcal{O}}_{X,x}$ generated by the ideal of $X \setminus U$ corresponds under this isomorphism to the ideal in $\hat{\mathcal{O}}_{S,s}$ generated by the ideal of $S \setminus T$, where T is the big torus of S.

We say that $U \subset X$ is a *toroidal embedding without self-intersection* if it is a toroidal embedding and every irreducible component of $X \setminus U$ is normal.

For a subset $I \subset \{0, \ldots, n+1\}$, we define

$$Z_I := \left(\bigcap\nolimits_{i \in \{0,\ldots,n+1\} \setminus I} (x_i = 0)\right) \bigcap \left(\bigcap\nolimits_{i \in I} (x_i \neq 0)\right) \bigcap X_{\mathrm{qs}},$$

where we write $x_{n+1} = y$. The toroidal embedding $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$ has a stratification by locally closed subsets $\{Z_I \mid I \subset \{0, \ldots, n+1\}\}$. For a subset $I \subset \{0, \ldots, n+1\}$, we define $r_I := \gcd\{a_i \mid i \in I\}$, where $a_{n+1} = b$. We put

$$\mathcal{I} := \{I \subset \{0, \ldots, n+1\} \mid r_I > 1, \text{ and if } r_I \mid d \text{ then } |I| > 1\}.$$

A stratum Z_I is contained in the singular locus of X_{qs} if and only if $I \in \mathcal{I}$, that is, we have

$$\mathrm{Sing}(X_{\mathrm{qs}}) = \bigcup\nolimits_{I \in \mathcal{I}} Z_I.$$

Corollary 2.5. *There exists a desingularization $\varphi \colon Y \to X$ with the following properties.*

1. *Around the singular points on $X \cap \mathrm{D}_+(x_0)$, φ is the composition of blow ups at each singular points.*
2. *The restriction $\varphi \colon \varphi^{-1}(X_{\mathrm{qs}}) \to X_{\mathrm{qs}}$ is a resolution of the toroidal embedding $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$.*

As a second step, we construct a big line bundle on a smooth model Y of X. There is a natural projection

$$\pi\colon \mathbb{P}(a_0, a_1, \ldots, a_n, b) \setminus \{(0 : \cdots : 0 : 1)\} \to \mathbb{P}(a_0, a_1, \ldots, a_n).$$

Let V be the smooth locus of $\mathbb{P}(a_0, a_1, \ldots, a_n)$. Put $U = \pi^{-1}(V)$ and $X^\circ = X \cap U$. By Condition (1.5.3), we see that U is smooth and the codimension of $X \setminus X^\circ$ in X is greater than or equal to 2. By a slight abuse of notation, the restriction of π on X° is again denoted by $\pi\colon X^\circ \to V$.

For an integer l, we denoted by $\mathcal{O}_{X^\circ}(l)$ the restriction of the tautological sheaf $\mathcal{O}(l)$ of $\mathbb{P}(a_0, a_1, \ldots, a_n, b)$ on X°. The sheaf $\mathcal{O}_{X^\circ}(l)$ is invertible on X° for every integer l since $\mathcal{O}(l)$ is invertible on U.

Lemma 2.6. *Notation as above.*

(1) *There is an exact sequence*; $0 \to \pi^*\Omega^1_V \to \Omega^1_U|_{X^\circ} \to \mathcal{O}_{X^\circ}(-b) \to 0$.
(2) *There is an exact sequence*; $0 \to \mathcal{O}_{X^\circ}(-d) \xrightarrow{\delta} \Omega^1_U|_{X^\circ} \to \Omega^1_{X^\circ} \to 0$, *and we have* $\operatorname{Im}\delta \subset \pi^*\Omega^1_V$.
(3) *There is an exact sequence*;

$$0 \to \operatorname{Coker}[\mathcal{O}_{X^\circ}(-d) \xrightarrow{\delta} \pi^*\Omega^1_V] \to \Omega^1_{X^\circ} \to \mathcal{O}_{X^\circ}(-b) \to 0.$$

Proof. There is a locally splitting exact sequence

$$0 \to \pi^*\Omega^1_V \to \Omega^1_U \to \mathcal{O}_U(-b) \to 0.$$

Pulling back this sequence to X° we obtain (1). The existence of the exact sequence of (2) is a general fact. (3) follows from (1) and (2). We check locally to see that $\operatorname{Im}\delta$ is contained in $\pi^*\Omega^1_V$.

Take a point $u \in X^\circ$. We can choose local coordinates $z_1, \ldots, z_n, w$ of U at u so that $z_1, \ldots, z_n$ form local coordinates of V at $\pi(u)$ and X° is defined by the equation $w^p g'(z_1, \ldots, z_n) - f'(z_1, \ldots, z_n) = 0$ around u, where g', f' and w correspond to x_0, f and y respectively. We see that $\operatorname{Im}\delta$ is generated by

$$d(w^p g' - f') = p w^{p-1} g' dw + w^p dg' - df' = w^p dg' - df'$$

and, thus, it is contained in $\pi^*\Omega^1_V$. □

Notice that X° is not smooth in general. It may have isolated singular points on $X^\circ \cap \mathrm{D}_+(x_0)$ as described in Lemma 2.2. If we restrict the sequences in (1), (2) and (3) of Lemma 2.6 on the smooth locus of X°, then those are exact sequences of locally free sheaves.

Definition 2.7. Let $\mathcal{M}^\circ$ be the double dual of

$$\bigwedge^{n-1} \left(\operatorname{Coker}[\mathcal{O}_{X^\circ}(-d) \xrightarrow{\delta} \pi^*\Omega^1_V]\right)$$

and $\mathcal{M} = i_*\mathcal{M}^\circ$, where $i\colon X^\circ \hookrightarrow X$ is the embedding.

Lemma 2.6 implies that

$$\mathcal{M} \cong \mathcal{O}_X\left(d - \sum\nolimits_{i=0}^{n} a_i\right) = \mathcal{O}_X(A),$$

and $\mathcal{M} \subset (\Omega_X^{n-1})^{\vee\vee}$. By Condition (1.5.4), M is ample.

In the following, we fix a desingularization $\varphi\colon Y \to X$ which satisfies properties (1) and (2) of Lemma 2.5. Let F be the exceptional divisor of φ which is obtained by resolving isolated singular points on $X \cap \mathrm{D}_+(x_0)$. Let E be the exceptional divisor of $\varphi\colon Y \to X$ away from F, that is, E is obtained by resolving the singularities of the toroidal embedding $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$ and then let $E = \cup_i E_i$ be the irreducible decomposition. Let M be a Weil divisor on X such that $\mathcal{O}_X(M) \cong \mathcal{M}$. The restriction of $\mathcal{O}_Y(\lfloor \varphi^* M \rfloor)$ on $Y \setminus (E \cup F)$ can be seen as a subsheaf of $\Omega_Y^{n-1}|_{Y \setminus (E \cup F)}$.

Definition 2.8. For each i, let γ_i be the largest integer such that $\mathcal{O}_Y(\lfloor \varphi^* M \rfloor + \gamma_i E_i)$ is contained in Ω_Y^{n-1} generically around E_i. We define $\mathcal{L} := \mathcal{O}_Y(\lfloor \varphi^* M \rfloor + \sum \gamma_i E_i)$.

By the definition, we have $\mathcal{L}|_{Y \setminus F} \subset \Omega_Y^{n-1}|_{Y \setminus F}$.

Lemma 2.9. *$\mathcal{L}$ is a subsheaf of Ω_Y^{n-1}.*

Proof. Put $X_0 = X \cap \mathrm{D}_+(x_0), Y_0 = \varphi^{-1}(X_0)$ and $\varphi_0 = \varphi|_{Y_0}\colon Y_0 \to X_0$. We need to show that $\mathcal{L}|_{Y_0} = \varphi_0^*(\mathcal{M}|_{X_0}) \subset \Omega_{Y_0}^{n-1}$. The restriction of the projection

$$\pi_0 = \pi|_{X_0}\colon X_0 \to \mathrm{D}_+(x_0) \subset \mathbb{P}(a_0, a_1, \dots, a_n)$$

is identified with the morphism

$$\operatorname{Spec} A[\nu]/(\nu^p - f') \to \operatorname{Spec} A = \mathbb{A}^n,$$

where $A = \Bbbk[\xi_1, \dots, \xi_n]$ and $f' = f(1, \xi_1, \dots, \xi_n) \in A$. Consider the homomorphism of A-modules $\rho_{f'}\colon A \to \Omega_A^1$ determined by $\rho_{f'}(1) = df'$. We have $\delta|_{X_0} = -\pi_0^* \rho_{f'}$ and this implies that $\mathcal{M}|_{X_0} = \pi_0^* \mathcal{Q}$, where $\mathcal{Q}$ is the invertible sheaf on $\mathbb{A}^n$ associated with the A-module $(\bigwedge^2 \operatorname{Coker}(\rho_{f'}))^{\vee\vee}$.

With the notation of [7], the invertible sheaf $\pi_0^* \mathcal{Q}$ is generated by a $(n-1)$-form η which is of type eta (cf. Remark 2.10 below and [7, Definition 22.3]) and $\varphi_0^* \eta$ does not have a pole along exceptional divisors of φ_0 (cf. [7, Section 22, 23]). Therefore, we have $\mathcal{L}|_{Y_0} = \varphi_0^*(\pi_0^* \mathcal{Q}) \subset \Omega_{Y_0}^{n-1}$. □

Remark 2.10. It is shown in [7, Lemma 16] that $\mathcal{M}|_{X \cap \mathrm{D}_+(x_0)} = \mathcal{O}_{X \cap \mathrm{D}_+(x_0)} \cdot \eta$, where

$$\eta = (\pm) \frac{d\xi_2 \wedge \cdots \wedge d\xi_n}{\frac{\partial}{\partial \xi_1}(\nu^p - f')} = (\pm) \frac{d\xi_1 \wedge d\xi_3 \wedge \cdots \wedge d\xi_n}{\frac{\partial}{\partial \xi_2}(\nu^p - f')} = \cdots = (\pm) \frac{d\xi_1 \wedge \cdots \wedge d\xi_{n-1}}{\frac{\partial}{\partial \xi_n}(\nu^p - f')},$$

is a $(n-1)$-form on X.

Let l be a sufficiently divisible positive integer so that $\mathcal{M}^{[l]}$ is an invertible sheaf on X, where $\mathcal{M}^{[l]}$ is the double dual of $\mathcal{M}^{\otimes l}$. Then, there are integers ε'_i such that

$$\mathcal{L}^{\otimes l} = \varphi^*\mathcal{M}^{[l]} \otimes \mathcal{O}_Y(-\sum_i \varepsilon'_i E_i).$$

Put $\varepsilon_i = \varepsilon'_i/l$. The rational number ε_i does not depend on the choice of l.

To conclude that the line bundle $\mathcal{L}$ is big, we need to lift global sections of $\mathcal{M}^{[l]}$ to those of $\mathcal{L}^{\otimes l}$. In other words, we need to bound the rational number ε_i from above.

Lemma 2.11. *Let E_i be an exceptional divisor of $\varphi\colon Y \to X$ whose center is $\bar{Z}_I$ for some $I \subset \{1, \dots, n+1\}$. Then, we have $A > r_I \varepsilon_i$.*

Proof. We give only a sketch of the proof. Let $x \in Z_I$ be a point. We can find a rational $(n-1)$-form ω_x on X such that $\mathcal{M}^{[l]} \subset \mathcal{O}_{X,x} \cdot \omega_x^{\otimes l}$. By the definition of ε_i, we see that ε_i is not greater than the order of the pole of $\varphi^*\omega_x$ along E_i.

Since $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$ is a toroidal embedding, there is a local model (S_I, s_I), which consists of an affine toric variety S_I and its point s_I, of X_{qs} at x. In particular, we have an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{S_I,s_I}$ of $\Bbbk$-local algebras. We can write down explicitly the rational $(n-1)$-form ω_I on S_I such that it corresponds to ω_x after passing to the completion. Let $\varphi_I\colon S' \to S_I$ be the resolution of toric varieties and E' the exceptional divisor of φ_I which corresponds formally to E_i. We can compute the order $\mathrm{ord}_{E'}(\varphi_I^*\omega_I)$ of the pole of $\varphi_I^*\omega_I$ along E'. Condition 1.6, which we omit, is nothing but the condition to ensure the inequality $A/r_I > \mathrm{ord}_{E'}(\varphi_I^*\omega_I)$. □

Lemma 2.12. *$\mathcal{L}$ is a big line bundle on Y.*

Proof. Put $a_{\max} = \max\{a_1, \dots, a_n\}$. By Lemma 2.11, we have $l\varepsilon_i \leq Al/r_I - a_{\max}$ for all sufficiently large and divisible l. We see that

$$\varphi_*\mathcal{L}^{\otimes l} = \mathcal{M}^{[l]} \otimes \varphi_*\mathcal{O}_Y\left(-\sum_i l\varepsilon_i E_i\right) \supset \mathcal{M}^{[l]} \otimes \varphi_*\mathcal{O}_Y\left(-\sum_{\varepsilon_i>0}(Al/r_I - a_{\max})E_i\right).$$

Consider the global sections $x_0^{Al}, x_0^{Al-a_1}x_1, \dots, x_0^{Al-a_n}x_n$ of $\mathcal{M}^{[l]} \cong \mathcal{O}_X(Al)$. Let U be a sufficiently small open subset of X such that $U \cap Z_I \neq \emptyset$. Then $x_0^{r_I}|_U \in \mathcal{O}_U$ and it vanishes along $\bar{Z}_I$. Hence, for each i, the section $x_0^{Al-a_i}x_i = (x_0^{r_I})^{Al/r_I - a_i}x_0^{(r_I-1)a_i}x_i$ vanishes along each singular stratum $\bar{Z}_I$ with multiplicity at least $Al/r_I - a_{\max}$ and thus lifts to a global section of $\mathcal{L}^{\otimes l}$.

The global sections $x_0^{Al}, \dots, x_0^{Al-a_n}x_n$ define a dominant map $X \dashrightarrow \mathbb{P}^n$. Therefore, $\mathcal{L}$ is big. □

As we explained in the beginning of this section, Theorem 1.8 follows from Lemma 2.9, 2.12 and 1.10.

3. Examples of nonrational $\mathbb{Q}$-Fano varieties

In this section, we present some examples of nonrational $\mathbb{Q}$-Fano varieties which are obtained by Theorem 1.7. When one looks for concrete examples, Condition (1.5.5) is not easy to use. The following gives a sufficient condition for it.

Lemma 3.1. *Let p be a prime number and $a_1, \dots, a_n, d$ positive integers with $p \nmid d$ and $n \geq 3$. Let $\Bbbk$ be an algebraically closed field of characteristic p and consider the polynomial ring $\Bbbk[x_1, \dots, x_n]$. Let Λ_d be the k-vector space $\Bbbk[x_1, \dots, x_n]_d$ of the weighted homogeneous polynomial of degree d, where the grading is given by $\deg(x_i) = a_i$. Assume that there exist positive integers $m_1, \dots, m_n$ such that at least one of the following holds*

(1) $\Lambda_d \supset \{x_1^{m_1}, \dots, x_n^{m_n}\}$,
(2) $\Lambda_d \supset \{x_1^{m_1}x_2,\ x_2^{m_2}, \dots, x_n^{m_n}\},\ p \nmid m_1$,
(3) $\Lambda_d \supset \{x_1^{m_1}x_2,\ x_2^{m_2}x_1,\ x_3^{m_3}, \dots, x_n^{m_n}\},\ p \nmid (m_1m_2 - 1)$.
(4) $\Lambda_d \supset \{x_1^{m_1}x_2,\ x_2^{m_2}x_3,\ x_3^{m_3}, \dots, x_n^{m_n}\},\ p \nmid m_1m_2$.
(5) $\Lambda_d \supset \{x_1^{m_1}x_3,\ x_2^{m_2}x_3,\ x_3^{m_3}, \dots, x_n^{m_n}\},\ p \nmid m_1m_2$.

Then, $\operatorname{Spec} \Bbbk[x_1, \dots, x_n]/(f)$ *is smooth outside the origin for a general $f \in \Lambda_d$.*

There are lists [5, 16.6], [1, Table 1] and [2, Table 1] of weighted hypersurfaces which are $\mathbb{Q}$-Fano threefolds with at most terminal singularities. We obtained Table 1 below by choosing members of those lists that satisfy both Condition 1.5 and 1.6. The integer c in Table 1 is defined as $c := -d + \sum_{i=0}^{n} a_i > 0$ so that we have $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(c)$. The singular points of X and types of singularities of X are written in the last column. P_i stands for the point $(0 : \cdots : 0 : 1 : 0 : \cdots : 0)$, where the 1 is in the i-th position, and P_{ij} is a point contained in the singular stratum $Z_{\{i,j\}}$. As a result, we obtain the following examples.

Theorem 3.2. *Let $p, a_0, \dots, a_3, b$ and d be integers listed in* Table 1. *Then, the weighted hypersurface*

$$X_f := (y^p x_0 - f(x_0, \dots, x_3) = 0) \subset \mathbb{P}_{\mathbb{C}}(a_0, a_1, a_2, a_3, b)$$

is a non-ruled $\mathbb{Q}$-Fano threefold with at most terminal singularities for a very general $f \in H_d(\mathbb{C})$.

Remark 3.3. [5, 16.6] (resp. [2, Table 1], [1, Table 1]) is the list of $\mathbb{Q}$-Fano weighted hypersurfaces of dimension three which have at most terminal singularities with $c = 1$ (resp. $c = 2, c \geq 3$).

It is proved in [3] that a general member of each of the 95 families listed in [5, 16.6] are (birationally) rigid, which implies the nonrationality of the general member. Thus, among the twelve families of our examples, No. 5 and 8 are new and the remaining ten provide the known cases with an alternate proof of nonrationality.

Let m, n be integers such that $4 \leq n, 0 < m < n$ and let l be an odd integer such that $n - m + 1 < l < 2(n - m)$. Then, for every odd positive integer a with $a > (m + 1)/2$, the combination

$$(p, a_0, \dots, a_m, a_{m+1}, \dots, a_n, b, n, d) = (2, 1, \dots, 1, a, \dots, a, (al - 1)/2, n, al)$$

satisfies Condition 1.5 and 1.6.

TABLE 1. A List of $(p, \{a_i\}, b, n, d)$ satisfying Condition 1.5 and 1.6.

	d	p	$(a_0, \ldots, a_3, b)$	c	Singularities
No. 1	5	2	$(1,1,1,1,2)$	1	$P_4 : \frac{1}{2}(1,1,1)$
No. 2	7	2	$(1,1,1,2,3)$	1	$P_3 : \frac{1}{2}(1,1,1), P_4 : \frac{1}{3}(1,1,2)$
No. 3	9	2	$(1,1,1,3,4)$	1	$P_4 : \frac{1}{4}(1,1,3)$
No. 4	10	3	$(1,1,1,5,3)$	1	$P_4 : \frac{1}{3}(1,1,2)$
No. 5	10	3	$(1,1,2,5,3)$	2	$P_4 : \frac{1}{3}(1,1,2)$
No. 6	15	2	$(1,1,2,5,7)$	1	$P_2 : \frac{1}{2}(1,1,1), P_4 : \frac{1}{7}(1,2,5)$
No. 7	15	2	$(1,1,3,4,7)$	1	$P_3 : \frac{1}{4}(1,1,3), P_4 : \frac{1}{7}(1,3,4)$
No. 8	15	2	$(1,2,3,5,7)$	3	$P_1 : \frac{1}{2}(1,1,1), P_4 : \frac{1}{7}(1,3,4)$
No. 9	16	3	$(1,1,2,8,5)$	1	$P_{23} : \frac{1}{2}(1,1,1), P_4 : \frac{1}{5}(1,2,3)$
No. 10	21	2	$(1,1,3,7,10)$	1	$P_4 : \frac{1}{10}(1,3,7)$
No. 11	22	3	$(1,1,3,11,7)$	1	$P_2 : \frac{1}{3}(1,1,2), P_4 : \frac{1}{7}(1,3,4)$
No. 12	28	3	$(1,1,4,14,9)$	1	$P_{23} : \frac{1}{2}(1,1,1), P_4 : \frac{1}{9}(1,4,5)$

Theorem 3.4. *Let m, n be integers such that $4 \leq n$ and $0 < m < n$, and let l be an odd positive integer such that $n - m + 1 < l < 2(n - m)$. Then, for every odd positive integer a with $a > (m + 1)/2$, the weighted hypersurface*

$$X_f := (y^2 x_0 - f(x_0, \ldots, x_n) = 0) \subset \mathbb{P}(\overbrace{1, \ldots, 1}^{m+1}, \overbrace{a, \ldots, a}^{n-m}, (al - 2)/2)$$

of degree al is a non-ruled log terminal $\mathbb{Q}$-Fano variety for a very general $f(x_0, \ldots, x_n) \in H_{al}(\mathbb{C})$.

Remark 3.5. The singular locus of X_f is the union of $\bar{Z}_{I_1} = (x_0 = \cdots = x_m = y = 0) \cap X_f$ and $Z_{I_2} = \{P_{n+1}\}$. The singularity of X_f at each point of $\bar{Z}_{I_1}$ is of type

$$\frac{1}{a}(\overbrace{1, \ldots, 1}^{m+1}, \overbrace{0, \ldots, 0}^{n-m-2}, b) = \frac{1}{a}(\overbrace{2, \ldots, 2}^{m+1}, \overbrace{0, \ldots, 0}^{n-m-2}, -1)$$

and that of X_f at P_{n+1} is of type

$$\frac{1}{b}(\overbrace{1, \ldots, 1}^{m}, \overbrace{a, \ldots, a}^{n-m}) = \frac{1}{b}(\overbrace{l, \ldots, l}^{m}, \overbrace{1, \ldots, 1}^{n-m}),$$

where $b = (al - 1)/2$.

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