代数幾何学シンポジウム記録
2009年度 pp.126-126

Classification of rank 2 weak Fano bundles on $\mathbb{P}^n$

Kazunori Yasutake

Abstract
In this poster, I classify weak Fano varieties which have a $\mathbb{P}^1$-bundle structure over $\mathbb{P}^n$ using vector bundle method.

Definition of weak Fano bundle

$\mathbf{X}$: smooth projective variety / $\mathbb{C}$,
$\mathcal{E}$: vector bundle on $\mathbf{X}$.
$\mathcal{E}$: weak Fano $\overset{\text{def}}{\iff}$ $\mathbf{Y} := \mathbb{P}_{\mathbf{X}}(\mathcal{E})$: weak Fano var.
(i.e. $-\mathbf{K_Y}$: nef and big)

Main Thm 1

$\mathcal{E}$: weak Fano bundle $\implies$ $\mathbf{X}$: weak Fano var.

Weak Fano bundle on $\mathbb{P}^n$

split case

Let $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(\mathbf{a_1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(\mathbf{a_r})$
: vector bundle on $\mathbb{P}^n$ s.t. $(\mathbf{0} \le \mathbf{a_1} \le \cdots \le \mathbf{a_r})$.
Then, $\mathcal{E}$: weak Fano $\iff$ $\mathbf{0} \le \sum_{i=1}^{r} \mathbf{a_i} \le \mathbf{n+1}$.

Main Thm 2

Let $\mathcal{E}$: normalized rank 2 weak Fano bundle on $\mathbb{P}^n$ $(\mathbf{n} \ge \mathbf{3})$.
(α) $\mathbf{n} \ge \mathbf{4}$ $\implies$ $\mathcal{E}$: split
(β) If $\mathbf{n} = \mathbf{3}$, then, $\mathcal{E}$ splits or
$(\beta 1)$ $\mathcal{E}$: stable, $\mathbf{c_1} = \mathbf{0}$, $\mathbf{c_2} = \mathbf{1}$ or
$(\beta 2)$ $\mathcal{E}$: stable, $\mathbf{c_1} = \mathbf{0}$, $\mathbf{c_2} = \mathbf{2}$ or
$(\beta 3)$ $\mathcal{E}$: stable, $\mathbf{c_1} = \mathbf{0}$, $\mathbf{c_2} = \mathbf{3}$.

Remark. There are weak Fano/non weak Fano 2-bundles on $\mathbb{P}^3$ satisfying the condition $(\beta 3)$.

First, we construct weak Fano 2-bundle satisfying $(\beta 3)$ using the theorem of Mori and the theory of regular bundle.

Theorem. [*M*] *There is a non-singular elliptic curve C on a smooth quartic surface $S \subset \mathbb{P}^3$ and a very ample divisor H on S s.t.*
(1) $Pic(S) \cong \mathbb{Z}[H] \oplus \mathbb{Z}[C]$.
(2) $H^2 = 4,\ H.C = 7,\ C^2 = 0$.
(3) *C is base point free.*

Example 1. Let (S, C): as above. Using the theory of elementary transformation, we can construct a rank 2 stable regular vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ with $c_1 \simeq S,\ c_2 \simeq C$. Then, we can show $\mathcal{F}$ is weak Fano. So $\mathcal{E} := \mathcal{F}(-2)$ is a weak Fano 2-bundle with $c_1 = 0,\ c_2 = 3$.

Next, we construct non weak Fano 2-bundle satisfying $(\beta 3)$ using well-known Serre construction.

Example 2. Let Y be a 4 disjoint union of lines in $\mathbb{P}^3$. By Serre construction, we can construct a rank 2 stable bundle $\mathcal{F}$ on $\mathbb{P}^3$ with $c_1 = 2, c_2 = 4$. Then, we can check $H^0(\mathcal{F}) \neq 0$. From easy computation, $\xi_{\mathcal{F}}.(-K_{\mathbb{P}(\mathcal{F})})^3 < 0$. So $\mathcal{E} := \mathcal{F}(-1)$ is non weak Fano stable bundle with $c_1 = 0$, $c_2 = 3$.

Main Thm 3

Let $\mathcal{E}$: normalized rank 2 weak Fano bundle on $\mathbb{P}^2$.
(γ) If $\mathbf{c_1}(\mathcal{E}) = -\mathbf{1}$, Then
$(\gamma 1)$ $\mathcal{E}$: not stable $\Rightarrow$ $\mathcal{E}$: split or $\mathcal{E}$ is determined by
$\mathbf{0} \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_{\mathbf{p}}(-\mathbf{1}) \to \mathbf{0}$.
$(\gamma 2)$ $\mathcal{E}$: stable $\Rightarrow$ $\mathcal{E} \cong \mathbf{T}_{\mathbb{P}^2}(-\mathbf{2})$ or $\mathbf{2} \le \mathbf{c_2}(\mathcal{E}) \le \mathbf{5}$
(δ) If $\mathbf{c_1}(\mathcal{E}) = \mathbf{0}$, Then
$(\delta 1)$ $\mathcal{E}$: not stable $\Rightarrow$ $\mathcal{E} \cong \mathcal{O}(-\mathbf{1}) \oplus \mathcal{O}(\mathbf{1})$.
$(\delta 2)$ $\mathcal{E}$: semistable and not stable
$\Rightarrow$ $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{E}$ is determined by
$\mathbf{0} \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_{\mathbf{p}} \to \mathbf{0}$.
$(\delta 3)$ $\mathcal{E}$: stable $\Rightarrow$ $\mathbf{2} \le \mathbf{c_2}(\mathcal{E}) \le \mathbf{6}$

Remark. The author does not know whether the case $(stable,\ c_1 = 0,\ c_2 = 6)$ in $(\delta 3)$ exists or not.

Known Result

[APW], [SW1], [SW2]

Let $\mathcal{E}$: normalized rank 2 Fano bundle on $\mathbb{P}^n$.
(α) $\mathbf{n} \ge \mathbf{4}$ $\implies$ $\mathcal{E}$: split
(β) If $\mathbf{n} = \mathbf{3}$, then, $\mathcal{E}$ splits or
$(\beta 1)$ $\mathcal{E}$: stable, $\mathbf{c_1} = \mathbf{0}$, $\mathbf{c_2} = \mathbf{1}$
(γ) If $\mathbf{n} = \mathbf{2}$, then,
$(\gamma 1)$ $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-\mathbf{1})$.
$(\gamma 2)$ $\mathcal{E} \cong \mathbf{T}_{\mathbb{P}^2}(-\mathbf{2})$.
$(\gamma 3)$ $\mathcal{E} \cong \mathcal{O}(-\mathbf{1}) \oplus \mathcal{O}(\mathbf{1})$.
$(\gamma 4)$ $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}$.
$(\gamma 5)$ $\mathbf{0} \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_{\mathbf{p}} \to \mathbf{0}$, $\mathbf{p} \in \mathbb{P}^2$.
$(\gamma 6)$ $\mathcal{E}$: stable, $\mathbf{c_1} = \mathbf{0}$, $\mathbf{c_2} = \mathbf{2}$.
$(\gamma 7)$ $\mathcal{E}$: stable, $\mathbf{c_1} = \mathbf{0}$, $\mathbf{c_2} = \mathbf{3}$.

References

[APW] V. Ancona, T. Peternell and J.A. Wisniewski, *Fano bundles and splitting theorems on projective spaces and quadrics*, Pacific J. Math. **163** (1994), no. 1, 17-42.

[SW1] M. Szurek and J.A. Wisniewski, *Fano bundles over* $\mathbb{P}^3$ *and* Q_3, Pacific J. Math. **141** (1990), 197-208.

[SW2] M. Szurek and J.A. Wisniewski, *Fano Bundles of Rank 2 on Surfaces*, Compositio Math. **76** (1990), 295-305.

[M] S. Mori, *On degrees and genera of curves on smooth quartic surfaces in* $\mathbb{P}^3$, Nagoya Math. J. **96** (1984), 127-132.