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Global monodromy modulo 5 of quintic-mirror family

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Abstract

When we fix a symplectic basis of the third cohomology group of a nonsingular fibre of quintic-mirror family, the global monodromy Γ is a subgroup of symplectic group $\mathrm{Sp}(4,\mathbb{Z})$. The generators of Γ are well studied by Candelas, de la Ossa, Green and Parks. We consider calculating components of matrixes which are elements of Γ under this situation. In main result, elements of a subgroup Γ' of $\mathrm{GL}(4,\mathbb{Z})$ which is isomorphic to Γ are represented by components modulo 5.

Quintic-mirror family

Let $(x_1 : \cdots : x_5)$ be the homogeneous coordinates of $\mathbb{P}^4$ and let $\psi \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We give a hypersurface Q_ψ of $\mathbb{P}^4$ by

$$Q_\psi := \{x \in \mathbb{P}^4 \mid \textstyle\sum_{i=1}^5 {x_i}^5 - 5\psi\Pi_{i=1}^5 x_i = 0\}.$$

A finite group G acts on Q_ψ as follows.

$$\mu_5 : \text{the group of 5-th root of } 1 \in \mathbb{C},$$

$$\widetilde{G} := (\mu_5)^5/\{(\alpha_1, \cdots, \alpha_5) \in (\mu_5)^5 \mid \alpha_1 = \cdots = \alpha_5\},$$

$$G := \{(\alpha_1, \cdots, \alpha_5) \in \widetilde{G} \mid \alpha_1 \cdots \alpha_5 = 1\},$$

$$G \times Q_\psi \to Q_\psi \ ;$$
$$((\alpha_1, \cdots, \alpha_5), (x_1 : \cdots : x_5)) \mapsto (\alpha_1 x_1 : \cdots : \alpha_5 x_5).$$

When we divide the hypersurface Q_ψ by G, canonical singularities appear. For $\psi \in \mathbb{C} \subset \mathbb{P}^1$, it is known that there is a simultaneous minimal desingularization of these singularities, and we have the family $(W_\psi)_{\psi\in\mathbb{P}^1}$ of the mirror to the above hypersurface. When ψ belongs to $\mu_5 \subset \mathbb{C} \subset \mathbb{P}^1$, W_ψ has one ordinary double point. W_∞ is a normal crossing divisor in the total space. The other fibres of $(W_\psi)_\psi$ are smooth with Hodge numbers $h^{p,q} = 1$ for $p+q = 3,\ p, q \geq 0$.

By the action of

$$\alpha \in \mu_5,\ (x_1, \cdots, x_5)\ \mapsto (x_1, \cdots, x_4, \alpha\ {}^1 x_5),$$

we have the isomorphism from the fibre over ψ to the fibre over $\alpha\psi$. Let $\lambda = \psi^5$ and let

$$\begin{array}{ccc} (W_\lambda)_\lambda & = & ((W_\psi)_\psi)/\mu_5 \\ \downarrow & & \downarrow \\ (\lambda\text{-plane}) & = & (\psi\text{-plane})/\mu_5. \end{array}$$

This family $(W_\lambda)_\lambda$ is so-called quintic-mirror family. Quintic-mirror family was constructed by Greene and Plesser.

Local monodromy

Let $b \in \mathbb{P}^1 - \{0, 1, \infty\}$. Candelas, de la Ossa, Green and Parks constructed a symplectic basis $\{A^1, A^2, B_1, B_2\}$ of $H_3(W_b, \mathbb{Z})$ and calculated the monodromies around $\lambda = 0, 1, \infty$ on the period integrals of a holomorphic 3-form on this basis. By the relation between the symplectic basis $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ of $H^3(W_b, \mathbb{Z})$ which is the dual basis of $\{B_1, B_2, A^1, A^2\}$ and the period integrals, we have the matrix representations of the local monodromies for the basis $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$. We recall their results.

Matrix representations A, T, T_∞ of local monodromies around $\lambda = 0, 1, \infty$ for the basis $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ are as follows:

$$A = \begin{pmatrix} 11 & 8 & -5 & 0 \\ 5 & -4 & -3 & 1 \\ 20 & 15 & -9 & 0 \\ 5 & -5 & -3 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_\infty = \begin{pmatrix} -9 & -3 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ -20 & -5 & 11 & 0 \\ -15 & 5 & 8 & 1 \end{pmatrix}.$$

Global monodromy

Let $\langle\ ,\ \rangle$ be the anti-symmetric bilinear form on $H^3(W_b, \mathbb{Z})$ defined by the cup product. Global monodromy Γ is $\mathrm{Im}\Big(\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to \mathrm{Aut}\big(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle\big)\Big)$.

When we take $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ as the basis $H^3(W_b, \mathbb{Z})$, $\mathrm{Aut}\big(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle\big)$ is identified $\mathrm{Sp}(4, \mathbb{Z})$, and Γ is the subgroup of $\mathrm{Sp}(4, \mathbb{Z})$ which is generated by A and T.

Lemma There exisits $P \in \mathrm{GL}(4, \mathbb{Q})$ such that

$$P\ {}^1 A\ {}^1 P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 5 & 5 & 5 & -4 \end{pmatrix}, P\ {}^1 T\ {}^1 P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\Gamma' := \{P\ {}^1 X P \in \mathrm{GL}(4, \mathbb{Z}) \mid X \in \Gamma\}$ is isomorphic to Γ as group.

Main result

Let $X \in \Gamma'$. Then, $X \equiv \begin{pmatrix} 1 & n & 3n^2 + 2n & a \\ 0 & 1 & n & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod 5$.

Here n, a, b, c is the elements of $\mathbb{Z}$.

References

[COGP] P. Candelas, C. de la Ossa, P. S. Green, and L. Parks, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, Nuclear Phys. B 359 (1991), no. 1, 21–74.

[M] D. Morrison, Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians, J. Amer. Math. Soc. 6 (1993), no. 1, 223–247.