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On the MWL of some elliptic $K3$ surface in char. 11

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1. Introduction.

We study the elliptic surface defined by the equation

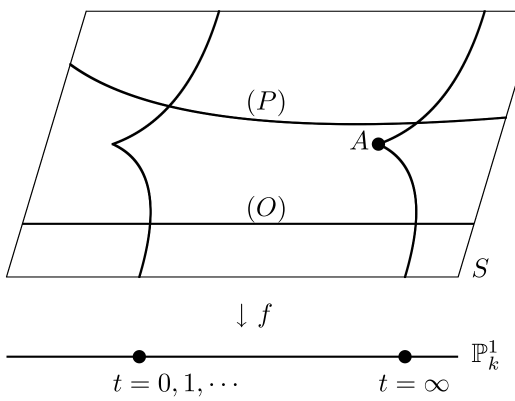
$$\mathbf{S : y^2 = x^3 + t^{11} - t \qquad /k = \overline{\mathbb{F}_{11}}.}$$

Why this equation? Why characteristic 11*?* These natural questions are closely related to the main problem of the field: *How can we construct an elliptic curve with bigger Mordell-Weil rank?* Our aim is to give a connection between the Mordell-Weil lattice of S and the Mathieu group via the theory of $K3$ lattices. We hope our result will help us to solve the questions such as

Problem 0.1 (Shioda)**.** What is the maximum of the number of *integral sections* for ellptic $K3$ surfaces?

2. Elliptic Surfaces and MWL.

An elliptic fibration of S is a morphism onto a nonsingular curve whose general fibers are elliptic curves.



Under some additional conditions, the group of sections $MW(f)$ is a finitely generated abelian group and it has a natural positive-definite quadratic form $\langle , \rangle$ (Shioda, Elkies).

Definition-Lemma 0.2. A section (P) is called *an integral section* if it is disjoint from the zero-section (O). For our S, this is equivalent for P to have the minimal norm as to $\langle , \rangle$.

3. Background.

Mukai (1988), Kondo (1998), Dolgachev and Keum (2009) have classified the finite symplectic automorphism groups of $K3$ surfaces. The classification for the case characteristic $p \leq 7$ is still open.

In another direction, Shioda (1991) applied the theory of Mordell-Weil lattices to the construction of denser sphere packings in many higher dimensions.

The surface S is the unique supersingular $K3$ surface of Artin invariant 1 in characteristic 11. In both theory, S and the automorphism group $\mathrm{GU}_2(11)$ appear as an important ingredient.

4. Main Theorem.

Theorem 0.3.

1. (A characterization of $MW(f)$): If a positive definite even lattice M is of rank 20, $A_M \simeq 11^2$, minimal norm ≥ 4 and have an automorphism of order 11 then $M \simeq MW(f)$. M is realized in the Niemeier lattice $L(A_1^{24})$ as the orthogonal complement of N, where

$$N = \left\langle e_1, \left(\sum_{i=1}^{12} e_i\right)/2, \left(\sum_{i=13}^{24} e_i\right)/2, e_{24} \right\rangle$$

   and $\{1, 2, \cdots, 12\}$ is a dodecad of the binary Golay code.

2. The kissing number of $MW(f)$ is $12,540$. The number of next length vectors is $252,792$.

3. S has a finite automorphism group G isomorphic to $PSL_2(11) : \mathbb{Z}/12\mathbb{Z}$ which preserve O. The orthogonal group of $MW(f)$ is generated by G and $\mathrm{Gal}(\mathbb{F}_{121}/\mathbb{F}_{11})$, which is a maximal finite subgroup of $\mathrm{GL}_{20}(\mathbb{Q})$. The splitting field of S is $\mathbb{F}_{121}$.

4. (A question of Dolgachev and Keum): $\mathrm{Aut}(S)$ contains infinitely many wild automorphisms of order 11 with an isolated fixed point.

Remark 0.4.

1. The kissing number has been known by a computer calculation. Our enumeration is group-theoretic.

2. The proof of 3. depends on the classification of maximal finite subgroups of $\mathrm{GL}_{20}(\mathbb{Q})$ due to Nebe and Plesken.

3. Dolgachev and Keum have shown that for $p \leq 7$ there exist wild $K3$ automorphisms of order p with an isolated fixed point, and for $p \geq 13$ there never exist. Thus $p = 11$ has been remaining.

5. Further Problems.

Does S give the maximum of the number of integral sections among elliptic $K3$ surfaces?

Can we classify the MWL of elliptic $K3$ surfaces via Niemeier lattices?

Are there any similar computation for surfaces with $= 1$?