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Classification of Rigid Analytic Surfaces

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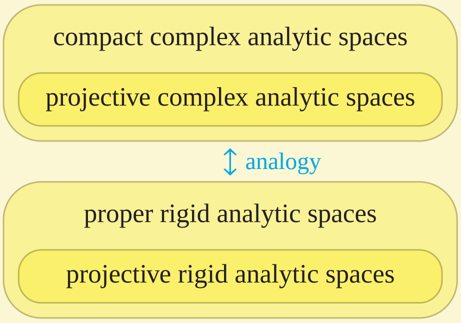
We classify rigid analytic surfaces. The classification is similar to the complex analytic case. In the positive characteristic case, new types of surfaces appear.

1. Rigid Analytic Geometry

Rigid analytic geometry is an analytic geometry over a complete non-Archimedean valuation field. We fix the base field K (i.e., $K = \mathbb{Q}_p$). The geometry is similar to complex analytic geometry:



Known facts:

- The *GAGA principle* holds for projective rigid analytic spaces.
- As in the complex analytic case, *any proper rigid analytic curve is projective*.

Problem. ***Classify proper rigid analytic surfaces.***

2. Complex Analytic Surfaces

Let us review fundamental theorems on the classification of compact complex analytic surfaces.

Theorem 1. *Any surface admits a* ***desingularization****.*

Theorem 2. *Any smooth surface admits a* ***minimal model****.*

Theorem 3. *A smooth surface is* ***minimal*** *if and only if it contains no* ***exceptional curves of the first kind****.*

Theorem 4. *Let $a(X)$ be the* ***algebraic dimension*** *of a smooth surface X. Then the followings hold:*

- $a(X) = 2 \Rightarrow$ *X is a* ***projective surface****.*
- $a(X) = 1 \Rightarrow$ *X is an* ***elliptic surface****.*

Here, we define $a(X)$ as the transcendental degree of the field of meromorphic functions on X over $\mathbb{C}$.

Theorem 5. *For minimal elliptic surfaces with Kodaira dimension $\kappa < 1$, we can list the* ***possible combinations of the invariants*** *(κ, g, χ, the multiplicities of the multiple fibers).* *Here, g is the genus of the base curve, and χ is the Euler characteristic of the structure sheaf of the surface.*

3. Algebraic Surfaces over a Base Field of Arbitrary Characteristic

Theorems 1–3 hold for proper regular algebraic surfaces over a base field of arbitrary characteristic. As for Theorem 5, when $K = \overline{K}$, *new types of surfaces* appear.

4. Rigid Analytic Surfaces

Main Result A. *Theorems 1–4 hold for proper rigid analytic surfaces. Here, we have to modify the statements: complex analytic → rigid analytic, compact → proper, smooth → regular, and in Theorem 4 we assume that the base field K is perfect.*

Main Result B. *As for Theorem 5, when $K = \overline{K}$, we can list the* ***combinations*** *(see Table).* ***New types of non-algebraic surfaces*** *appear. We show the canonical bundle formula and Noether's formula for elliptic surfaces.*

Main Result C. *If an elliptic fibration admits a relative Tate uniformization, we can define* ***logarithmic transformation****. The logarithmic transformation gives examples of* ***non-algebraic surfaces*** *and* ***algebraic surfaces of unknown types****.*

κ	χ	g	l	char.K	multiple fibers $(a_1/m_1, \ldots, a_n/m_n)$
$-\infty$	0	0	0		none, $(m-1/m)$
					$(m_1-1/m_1, m_2-1/m_2)$
					$(1/2, 1/2, m-1/m)$
$-\infty$	0	0	1	$\neq 0$	(a/m^*), $(a/m_1^*, m_2-1/m_2)$
$-\infty$	1	0	0		none, $(m-1/m)$
0	0	0	0		$(1/2, 2/3, 5/6)$, $(1/2, 3/4, 3/4)$
					$(2/3, 2/3, 2/3)$, $(1/2, 1/2, 1/2, 1/2)$
0	0	0	1	2	$(0/2^*, 1/2, 1/2)$
					$(1/2^*, 1/2)$, $(1/4^*, 3/4)$
					$(2/4^*, 1/2)$, $(2/6^*, 2/3)$
0	0	0	1	3	$(1/3^*, 2/3)$, $(3/6^*, 1/2)$
0	0	0	2	$\neq 0$	$(0/p^{a*})$, $(0/p^{a*}, 0/p^{b*})$
0	0	1	0		none
0	1	0	0		$(1/2, 1/2)$
0	1	0	1	2	$(0/2^*)$
0	2	0	0		none

Table. The possible invariants of minimal elliptic surfaces $\pi\colon X \to S$ with $\kappa < 1$. The integer m_i is the multiplicity of the i-th multiple fiber. The symbol $*$ means that the multiple fiber is wildly ramified. $\mathcal{K}_X = \pi^*\mathcal{L} \otimes \mathcal{O}_X(\sum_i a_i D_i)$, $\deg \mathcal{L} = \chi + 2g - 2 + l$, and l = length torsion $R^1\pi_*\mathcal{O}_X$.

5. Proofs

We use the following techniques:

- *Local algebraization.* For any proper morphism $\phi : X \to Y$ of relative dimension at most one and any point $y \in Y$, there exists an "open neighborhood" U of y such that the restriction $\phi|_{\phi^{-1}(U)}$ is the analytification of a morphism of schemes. ⇒ Theorem 1, 2, and 4 in Main Result A
- *Deformation theory.* Non-Archimedean analysis enables us to deform divisors whenever the obstructions vanish. ⇒ Theorem 3 in Main Result A
- *Good étale covering.* We construct "good" étale covering that relates rigid analytic geometry to scheme theory. The base change of a fibration via the covering is locally the analytification of a morphism of schemes. ⇒ Main Result B
- *Tate's uniformization.* We use Tate's elliptic functions to trivialize certain elliptic fibrations. ⇒ Main Result C

6. Problems

- ***Further classification.*** *Elliptic surfaces. Surfaces with $a(X) = 0$.*
- ***Hodge theory.*** *What is Kähler?*
- ***Index theorem.*** *The Riemann-Roch theorem.*