

Defining ideal of the Segre locus in arbitrary characteristic

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$X \subset \mathbb{P}^N$: a non-deg proj variety over $k = \bar{k}$ with $p = \text{char}(k) \geq 0$.

Def (Segre locus). $\pi_z : \mathbb{P}^N \setminus \{z\} \rightarrow \mathbb{P}^{N-1}$: the proj from a pt $z \in \mathbb{P}^N$.
 $\mathfrak{S}^{\text{out}}(X) := \{z \in \mathbb{P}^N \setminus X \mid \pi_{z|X} : X \rightarrow \pi_z(X) \text{ is not birational}\}^-$,
 $\mathfrak{S}^{\text{inn}}(X) := \{z \in X \mid \pi_{z|X} : X \setminus \{z\} \rightarrow \pi_z(X) \text{ is not birational}\}^-$.

Theorem A (B. Segre, 1936). Assume $p = 0$. Then,
 $\mathfrak{S}^{\text{out}}(X) = (\text{finitely many linear subspaces in } \mathbb{P}^N)$.

- Recently, $\mathfrak{S}^{\text{out}}(X)$, $\mathfrak{S}^{\text{inn}}(X)$, and related topics have been studied by A. Calabri-C. Ciliberto, E. Ballico, A. Noma, ...

Q. Does the **linearity** of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$ hold in **char $p \geq 0$** ?

Theorem 1 (F). Assume $p \geq \text{deg}(X)$ or $p = 0$. Then,
 $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X) = (\text{finitely many linear subspaces in } \mathbb{P}^N)$.

Example 2 ($p < \text{deg}(X)$, Fukasawa-F). ℓ : a prime number,
 $X := (x_0^\ell - x_1 x_2^{\ell-1} = x_1^\ell - x_2 x_3^{\ell-1} = 0) \subset \mathbb{P}^3$ (a rational curve of $\text{deg } \ell^2$).

1. Either $p \neq \ell$ or $p = 0 \implies \mathfrak{S}^{\text{out}}(X) = \{(1, 0, 0, 0)\}$,
 Moreover, $\ell \geq 3 \implies \mathfrak{S}^{\text{inn}}(X) = \{(0, 0, 0, 1)\}$.
2. $p = \ell \implies \mathfrak{S}^{\text{out}}(X) = (x_3 = x_0^\ell x_2 - x_1^{\ell+1} = 0)$, **(non-linear)**
 Moreover, $p = \ell \geq 3 \implies \mathfrak{S}^{\text{inn}}(X) = X$.
3. $\ell = 2$ & $p \geq 0 \implies \mathfrak{S}^{\text{inn}}(X) = \emptyset$.

Rem. Linearity of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X) \implies$ linearity of $\mathfrak{S}^{\text{out}}(X)$.
 $(\because \forall \text{ irred comp } Z \subset \mathfrak{S}^{\text{out}}(X), Z = (\text{an irred comp of } \mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)).)$

Rem. The following is an essential fact in $p = 0$:

For a variety $Y \subset \mathbb{P}^N$, and for a linear subspace $L \subset \mathbb{P}^N$,
 $(\diamond) \quad \dim \pi_L(Y \setminus L) = \dim(Y) - \dim(\mathbb{T}_y Y \cap L) - 1$ ($y \in Y$: general pt).

In $p > 0$, equality (\diamond) does *not* hold in general, since the **separability of the map π_L** is *not* guaranteed.

Our strategy to prove Thm 1 & Ex 2.

Assume $p \geq 0$ & $\text{codim}(X, \mathbb{P}^N) = 2$.
 (\star) Give a method to compute the defining ideal of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$.

Assume $p \geq \text{deg}(X)$ & $\text{codim}(X, \mathbb{P}^N) = 2$.
 Show the **linearity** of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$.

By **induction** on $\text{codim}(X, \mathbb{P}^n)$, show the **linearity** of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$. **(Thm 1)**

For X in Ex 2, calc the ideal of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$.
 \leadsto The **non-linearity** is shown. **(Ex 2)**

Defining ideal of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$.

X : not a cone & $\text{codim}(X, \mathbb{P}^N) = 2$. For $e \in \mathbb{N}$,

$$\text{Loc}_e(X) := \{z \in \mathbb{P}^N \mid \text{deg}(\pi_z(X \setminus \{z\}))^- \leq e\}.$$

Prop 3. For every irred comp Z of $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$,

$\exists e < \text{deg}(X)$ s.t.

$Z = (\text{an irred comp of } \text{Loc}_e(X))$.

Thm 4. $r := h^0(\mathbb{P}^N, \mathcal{I}_X(e))$. Then,

$$\text{Loc}_e(X) = \left(\begin{array}{c} \text{the zero set of all} \\ r \times r \text{ minors of } \Lambda(e) \end{array} \right),$$

where $\Lambda(e)$ is constructed as follows.

Construction of the matrix $\Lambda(e)$.

$\mathbb{P}^N = \{(x_0, x_1, \dots, x_N)\}$,
 $i = (i_0, i_1, \dots, i_N) \in \mathbb{Z}_{\geq 0}^{N+1}$.

For $0 \leq k \leq N$, k
 $\omega_k := (0, \dots, 0, \overset{\cdot}{1}, 0, \dots, 0)$.

For each i ,

$$D_i := \frac{1}{i_0! i_1! \cdots i_N!} \left(\frac{\partial}{\partial x_0} \right)^{i_0} \left(\frac{\partial}{\partial x_1} \right)^{i_1} \cdots \left(\frac{\partial}{\partial x_N} \right)^{i_N}$$

(iterative higher derivation).

Def 5. $\{h_1, \dots, h_r\}$: a basis of $H^0(\mathbb{P}^N, \mathcal{I}_X(e))$. For an integer $s \leq e$,

$$\Lambda_{e-s} := \begin{bmatrix} D_{(e-s)\omega_0} h_1 & D_{(e-s-1)\omega_0+\omega_1} h_1 & \cdots & D_i h_1 & \cdots & D_{(e-s)\omega_N} h_1 \\ D_{(e-s)\omega_0} h_2 & D_{(e-s-1)\omega_0+\omega_1} h_2 & \cdots & D_i h_2 & \cdots & D_{(e-s)\omega_N} h_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{(e-s)\omega_0} h_r & D_{(e-s-1)\omega_0+\omega_1} h_r & \cdots & D_i h_r & \cdots & D_{(e-s)\omega_N} h_r \end{bmatrix}_{(|i|=e-s)}.$$

$$\Lambda(e) := \begin{cases} \Lambda_{e-1} & (p > e \text{ or } p = 0), \\ [\Lambda_{e-1} \ \Lambda_{e-p} \ \Lambda_{e-2p} \ \cdots \ \Lambda_{e-\lfloor e/p \rfloor p}] & (p \leq e). \end{cases}$$