代数幾何学シンポジウム記録
2009年度　pp.113-113

Key varieties and surfaces of general type

Stephen Coughlan, stephencoughlan21@gmail.com

World Class University Project, Mathematics Department, Sogang University, Seoul, Korea

Introduction

A surface of general type is a nonsingular surface S with K_S nef and big. The canonical model X of S is defined by

$$X = \mathrm{Proj} \bigoplus_{n\geq 0} H^0(S, nK_S),$$

and X may have rational double points arising from contracted (-2)-curves on S.

We construct a component of the moduli space of surfaces of general type with $p_g = 1$, $q = 0$ and $K^2 = 2$.

The canonical model for these surfaces is codimension 5, and so the commutative algebra lurking in the background is quite complicated.

A construction for these surfaces was given in [1], but it does not seem to be very easy to apply [1] to examples.

Fortunately we are able to simplify matters using the key varieties technique.

Symmetric determinantal K3 surfaces

Consider the quartic hypersurface $T_4 \subset \mathbb{P}^3$ defined by

$$\det M = 0,$$

where M is a symmetric matrix with linear entries in the coordinate variables y_i.

In general T is a K3 surface, and has ten nodes at points where the rank of M drops to 2.

Moreover by considering the cokernel of M, we obtain an ineffective Weil divisor A such that $\mathcal{O}_T(A)^{[2]} = \mathcal{O}_T(1)$:

$$0 \leftarrow \mathcal{O}_T(A) \leftarrow 4\mathcal{O}_{\mathbb{P}^3}(-1) \xleftarrow{M} 4\mathcal{O}_{\mathbb{P}^3}(-2) \leftarrow 0.$$

This gives rise to the surface $T \subset \mathbb{P}(2,2,2,2,3,3,3,3)$ in weighted projective space, defined by the equations

$$zM = 0, \quad z_i z_j = M_{ij}$$

where the variables y_i now have weighted degree 2, z_i are of weighted degree 3, and M_{ij} denotes the (i,j)th cofactor of M.

This is a K3 surface with $10 \times \frac{1}{2}(1,1)$ points where the rank of M drops to 2.

If we take a hypersurface section of weight 2 of T then we obtain a curve of genus 3 polarised by an ineffective theta characteristic.

Projection–unprojection construction

Alternatively, we can construct $T \subset \mathbb{P}(2,2,2,2,3,3,3,3)$ using a projection argument.

The projection from a $\frac{1}{2}$ point P in T is the following factored birational map

$$\begin{array}{ccccc} & & E \subset \widetilde{T} & & \\ & \swarrow^{\sigma} & & \searrow^{\varphi} & \\ \mathbb{P}(2,2,2,2,3,3,3,3) \supset T \ni P & & \overset{\pi}{\dashrightarrow} & & \mathbb{P}^1 \subset \bar{T}_{6,6} \subset \mathbb{P}(2,2,2,3,3) \end{array}$$

where σ is the weighted blowup of P, E is the exceptional locus of the blowup, and φ is determined by the linear system $\sigma^*(A) - \frac{1}{2}E$.

The projected surface $\bar{T}$ is a double covering of the plane $\mathbb{P}(2,2,2)$ branched in a sextic curve, and the rational curve $\mathbb{P}^1 \subset \bar{T}$ is the image of E under φ.

The branch sextic on $\bar{T}$ breaks into two cubic curves, and the rational curve is totally tangent to the two cubics in the plane. (Take a look at the figure on the right.)

This special configuration of curves in the plane is equivalent to a construction of the K3 surface $T \subset \mathbb{P}(2,2,2,2,3,3,3,3)$ via unprojection.

For more details on projection and unprojection, see the online database [4], and [5].

Extensions of K3 surfaces – the key variety

The K3 surface T can be realised as the anticanonical divisor inside a Fano 3-fold. In fact, we can extend T to a Fano 6-fold:

$$T \subset W \subset \mathbb{P}(1,1,1,1,2,2,2,2,3,3,3,3),$$

where W has $10 \times \frac{1}{2}$ points.

The easiest way to do this is to extend the projected surface $\bar{T}$ to a 6-fold $\bar{W}$ while preserving the totally tangent conic.

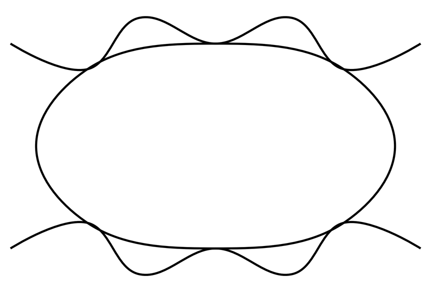
Remarkably, this configuration is so special that we obtain a one-to-one correspondence in moduli between the K3 surface and the Fano 6-fold:

Theorem [2] For each quasismooth symmetric determinantal K3 surface $T \subset \mathbb{P}(2^4, 3^4)$ with $10 \times \frac{1}{2}$ points there is a unique extension to a quasismooth Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4)$ with $10 \times \frac{1}{2}$ orbifold points and such that

$$T = W \cap H_1 \cap H_2 \cap H_3 \cap H_4,$$

where the H_i are hyperplanes of the projective space $\mathbb{P}(1^4, 2^4, 3^4)$.

Totally tangent conic



Surfaces of general type

We can use the key variety W to construct a family of surfaces of general type with $p_g = 1$, $q = 0$ and $K^2 = 2$ by intersecting with appropriate weighted hyperplane sections

Theorem [2] There is a 16 parameter family of surfaces of general type with $p_g = 1$, $q = 0$ and $K^2 = 2$ and no torsion, each of which is a complete intersection of type $(1,1,1,2)$ in a Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4)$ with $10 \times \frac{1}{2}$ points.

The surfaces we construct are the canonical models, and this family is believed to be an irreducible component of the moduli space of such surfaces.

The key variety method can be applied in various other situations, including

- There is a hyperelliptic degeneration which gives constructions for surfaces of general type with hyperelliptic canonical curve, see [3].
- There is a subfamily of surfaces with a fixed point free $\mathbb{Z}/2$ group action on each surface. The quotient is the general Godeaux surface with $\pi_1 = \mathbb{Z}/2$, see [3].

References

[1] F. Catanese, O. Debarre, Surfaces with $K^2 = 2$, $p_g = 1$, $q = 0$, Jour. reine. angew. 395 (1989) 1–55

[2] S. Coughlan, Extending symmetric determinantal quartic surfaces, arXiv:0909.5547

[3] S. Coughlan, Extending hyperelliptic K3 surfaces, and Godeaux surfaces with torsion $\mathbb{Z}/2$, arXiv:0909.5548

[4] G. Brown, Graded ring database, http://grdb.lboro.ac.uk — previous version at http://malham.kent.ac.uk/grdb

[5] M. Reid, Graded rings and birational geometry, Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 172