<table>
<thead>
<tr>
<th>Title</th>
<th>Key varieties and surfaces of general type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Coughlan, Stephen</td>
</tr>
<tr>
<td>Citation</td>
<td>代数幾何学シンポジウム記録 (2009), 2009: 113-113</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/214900">http://hdl.handle.net/2433/214900</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
A surface of general type is a non-singular surface \( S \) with \( K_S \) nef and big. The canonical model \( X \) of \( S \) is defined by

\[
X = \text{Proj} \bigoplus_{n \geq 0} \mathcal{H}^0(S, nK_S),
\]

and \( X \) may have rational double points arising from contracted \((-2)\)-curves on \( S \).

We construct a component of the moduli space of surfaces of general type with \( p_g = 1 \), \( q = 0 \) and \( K^2 = 2 \).

The canonical model for these surfaces is codimension 5, and so the commutative algebra lurking in the background is quite complicated.

A construction for these surfaces was given in [1], but it does not seem to be very easy to apply to examples.

Fortunately we are able to simplify matters using the key variety technique.

### Symmetric determinantal K3 surfaces

Consider the quartic hypersurface \( T_4 \subset \mathbb{P}^4 \) defined by

\[
\det(M) = 0,
\]

where \( M \) is a symmetric matrix with linear entries in the coordinate variables \( x_i \).

In general \( T \) is a K3 surface, and has ten nodes at points where the rank of \( M \) drops to 2.

Moreover by considering the cokernel of \( M \), one finds that it is a K3 surface, and has ten nodes at points where the rank of \( M \) drops to 2.

The K3 surface \( T \) is defined by the equations

\[
2 \cdot 2 = 0, \quad a_{ij} = M_{ij}
\]

where the variables \( a_{ij} \) have weight \( 2 \), \( a_{ij} \) are of degree 3, and \( M_{ij} \) denotes the \((i,j)\)th cofactor of \( M \).

This is a K3 surface with 10 \( \times 10 \) points where the rank of \( M \) drops to 2.

If we take a hypersurface section of weight 2 of \( T \) then we obtain a curve of genus 3 polarised by an ineffective theta characteristic.

### Extension of K3 surfaces - the key variety

The K3 surface \( T \) can be realised as the anticanonical divisor inside a Fano 3-fold. In fact, we can extend \( T \) to a Fano 6-fold:

\[
T \subset W \subset \mathbb{P}(1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3),
\]

where \( W \) has 10 \( \times 2 \) points.

The easiest way to do this is to extend the projected surface \( T \) to a 6-fold \( W \) while preserving the totally tangent cone.

Remarkably, this configuration is so special that we obtain a one-to-one correspondence in moduli between the K3 surface and the Fano 6-fold.

### Projection/unprojection construction

Alternatively, we can construct \( T \subset \mathbb{P}(2, 2, 2, 2, 3, 3, 3, 3) \) using a projection argument.

The projection from a \( \frac{1}{2} \) point \( P \) in \( T \) is the following factored birational map

\[
E \subset \hat{T}
\]

where \( E \) is the weighted blowup of \( P \), \( \hat{T} \) is the exceptional locus of the blowup, and \( \varphi \) is determined by the linear system \( \sigma(A) = K \).

The projected surface \( \hat{T} \) is a double covering of the plane \( \mathbb{P}(2, 2, 2) \) branched at a sextic curve, and the canonical curve \( \mathbb{P}^3 \subset \mathbb{P} \) is the image of \( \hat{T} \) under \( \varphi \).

The branch sextic of \( \hat{T} \) breaks into two cubic curves, and the rational curve is totally tangent to the two curves in the plane. (Take a look at the figure on the right.)

This special configuration of curves in the plane is equivalent to a construction of the K3 surface \( T \subset \mathbb{P}(2, 2, 2, 3, 3, 3, 3, 3) \) via unprojection.

For more details on projection and unprojection, see the online database [4], and [5].

### Totally tangent cone

We can use the key variety \( W \) to construct a family of surfaces of general type with \( p_g = 1 \), \( q = 0 \) and \( K^2 = 2 \) by intersecting with appropriate weighted hyperplane sections.

**Theorem 2** There is a 16 parameter family of surfaces of general type with \( p_g = 1 \), \( q = 0 \) and \( K^2 = 2 \) and no torsion, each of which is a complete intersection of type \((1, 1, 2, 2)\) in a Fano 6-fold \( W \subset \mathbb{P}(1, 1, 2, 2, 2, 3) \) with 10 \( \times 2 \) points.

The surfaces we construct are the canonical models, and this family is believed to be an irreducible component of the moduli space of such surfaces.

The key variety method can be applied in various other situations, including:

- There is a hyperelliptic degeneration which gives constructions for surfaces of general type with hyperelliptic canonical curve, see [6].
- There is a subfamily of surfaces with a fixed point free \( Z/2 \) group action on each surface. The quotient is the general Godeaux surface with \( \sigma = 2 \); see [5].

### References

[1] F. Catanese, O. Debarre, Surfaces with \( K^2 = 2 \), \( p_g = 1 \), \( q = 0 \), Jour. reine angew. Math. 393 (1989) 1–55