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<th>Key varieties and surfaces of general type</th>
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A surface of general type is a nonsingular surface $S$ with $K_S$ nef and big. The canonical model $X$ of $S$ is defined by

$$X = \text{Proj} \bigoplus_{n=0}^\infty H^0(S, nK_S),$$

and $X$ may have rational double points arising from contracted $(-2)$-curves on $S$.

We construct a component of the moduli space of surfaces of general type with $p_g = 1$, $q = 0$ and $K^2 = 2$.

The canonical model for these surfaces is codimension 5, and so the commutative algebra linking in the background is quite complicated.

A construction for these surfaces was given in [1], but it does not seem to be very easy to apply [1] to examples.

Fortunately we are able to simplify matters using the key variety technique.

### Symmetric determinantal K3 surfaces

Consider the quartic hypersurface $T_4 \subset \mathbb{P}^3$ defined by

$$\det M = 0,$$

where $M$ is a symmetric matrix with linear entries in the coordinate variables $y_i$.

In general $T_4$ is a K3 surface, and has ten nodes at points where the rank of $M$ drops to 2.

Moreover by considering the cokernel of $M$, we obtain an ineffective Weil divisor $A$ such that $\mathcal{O}_T(A)^{\otimes 2} = \mathcal{O}_T(1)$.

$$0 - \mathcal{O}_T(A) = 4\mathcal{O}_T(-1) = \mathcal{O}_T(-2).$$

This gives rise to the surface $T \subset \mathbb{P}(2, 2, 2, 3, 3, 3, 3)$ in weighted projective space, defined by the equations

$$z_0M = 0, \quad z_0z_j = M_{ij}$$

where the variables $y_i$ now have weight degree 2, $z_i$ are of weight degree 3, and $M_{ij}$ denotes the $(i, j)$th cofactor of $M$.

This is a K3 surface with $10 \times \frac{1}{2} = 5$ points where the rank of $M$ drops to 2.

If we take a hypersurface section of weight 2 of $T$ then we obtain a curve of genus 3 polarised by an ineffective theta characteristic.

### Projection-unprojection construction

Alternatively, we can construct $T \subset \mathbb{P}(2, 2, 2, 3, 3, 3, 3)$ using a projection argument.

The projection from a $\frac{1}{2}$ point $P$ of $T$ is the following factorised birational map

$$E \subset \tilde{T} \longrightarrow \tilde{P} \longrightarrow \mathbb{P}(2, 2, 2, 3, 3, 3, 3) \subset \mathbb{P}(2, 2, 2, 3, 3, 3),$$

where $\sigma$ is the weighted blowup of $P$, $E$ is the exceptional locus of the blowup, and $\varphi$ is determined by the linear system $\sigma^*(A) = 1E$.

The projected surface $\tilde{T}$ is a double covering of the plane $\mathbb{P}(2, 2, 2)$ branched in a sextic curve, and the rational curve $\tilde{P} \subset \tilde{T}$ is the image of $E$ under $\varphi$.

The branch sextic on $\tilde{T}$ breaks into two cubic curves, and the rational curve is totally tangent to the two cubics in the plane. (Take a look at the figure on the right.)

This special configuration of curves in the plane is equivalent to a construction of the K3 surface $T \subset \mathbb{P}(2, 2, 2, 3, 3, 3, 3)$ via unprojection.

For more details on projection and unprojection, see the online database [4] and [5].

### Total tangent conic

We can use the key variety $W$ to construct a family of surfaces of general type with $p_g = 1$, $q = 0$ and $K^2 = 2$ by intersecting with appropriate weighted hyperplane sections.

**Theorem** [2] There is a 16 parameter family of surfaces of general type with $p_g = 1$, $q = 0$ and $K^2 = 2$ and no torsion, each of which is a complete intersection of type $(1, 1, 1, 2)$ in a Fano 6-fold $W \subset \mathbb{P}(1^4, 2^3)$ with $10 \times \frac{1}{2}$ points.

The surfaces we construct are the canonical models, and this family is believed to be an irreducible component of the moduli space of such surfaces.

The key variety method can be applied in various other situations, including

- There is a hyperelliptic degeneration which gives constructions for surfaces of general type with hyperelliptic canonical curve, see [6].

- There is a subfamily of surfaces with a fixed point free $\mathbb{Z}/2$ group action on each surface. The quotient is the general Godeaux surface with $p_g = 2/2$ in [6].

### References


