Resolution of dihedral orbifolds

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Resolution of dihedral orbifolds

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Let \( G \subset \text{GL}(2, \mathbb{C}) \) be the following small binary dihedral group:

\[
G = \left\langle \alpha = \left( \begin{array}{cc} \varepsilon & 0 \\ 0 & \varepsilon^{-a} \end{array} \right), \beta = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) : \varepsilon^{2n} = 1, (2n, a) = 1, a^2 \equiv 1 \pmod{2n} \right\rangle
\]

where \( A = \langle \alpha \rangle = \frac{1}{2n}(1, a) \) is a maximal normal of index 2, and we consider the minimal resolution \( Y \rightarrow \mathbb{C}^2/G \).

\[
Y = \text{G-Hilb}(\mathbb{C}^2) : \text{Moduli space parametrising G-clusters ([Ishii])}
\]

\[
= \mathcal{M}_\theta(Q, R) : \text{Moduli of } \theta\text{-stable representations of the bound McKay quiver}
\]

**Definition:** Let \( G \subset \text{GL}(2, \mathbb{C}) \) be a finite subgroup. A **G-graph** is a subset \( \Gamma \subset \mathbb{C}[x, y] \) such that it contains \( \text{dim } \rho \) elements in each irreducible representation \( \rho \).

**Motivation:** For any \( G \)-cluster \( Z \in \text{G-Hilb}(\mathbb{C}^2) \), the basis of \( \mathcal{O}_Z \) as a vector space is a G-graph. Given a G-graph \( \Gamma \), all the G-clusters with \( \Gamma \) as basis for \( \mathcal{O}_Z \) form an open set \( U_\Gamma \subset \text{G-Hilb}(\mathbb{C}^2) \), and the collection of distinguished \( \{ U_\Gamma \}_{\Gamma} \) covers G-Hilb(\mathbb{C}^2).

**General construction**

- **Fact:** \( \text{G-Hilb}(\mathbb{C}^2) = G/A\text{-Hilb}(A\text{-Hilb}(\mathbb{C}^2)) \).
- The symmetry of the continued fraction \( \frac{2n}{a} \) implies that (i) the coordinates along the exceptional divisor \( E \subset A\text{-Hilb}(\mathbb{C}^2) \) are also symmetric, and (ii) \( \beta \) is an involution on the middle curve \( E_m \cong \mathbb{P}^1 \) on the exceptional divisor on \( A\text{-Hilb}(\mathbb{C}^2) \).
- Every G-graph \( \Gamma \) is either the unique extension of the union of two symmetric A-graphs, or it comes from a choice on the special irreducible representations \( \rho_\beta^+ \) and \( \rho_\beta^- \).

\[
\langle \beta \rangle \subset A\text{-Hilb}(\mathbb{C}^2) \Rightarrow \mathbb{C}^2/A
\]

\[
\begin{array}{c}
\Gamma \\
\rho_\beta^+ \rho_\beta^- \\
G\text{-Hilb}(\mathbb{C}^2) \\
A\text{-Hilb}(\mathbb{C}^2)/\langle \beta \rangle
\end{array}
\]

\[
\begin{array}{c}
\uparrow \\
(1, 1) \times 2 \text{ singularities}
\end{array}
\]

**Orbifold McKay quiver**

Let \( \text{Irr } A = \{ \rho_0, \ldots, \rho_{2n-1} \} \) the irreducible representations of \( A \). The McKay quiver of \( A = \frac{1}{2n}(1, a) \) can be written on a torus, and the quotient \( G/A \cong \mathbb{Z}/2 = \langle \beta \rangle \) acts on \( \text{Irr } A \) by conjugation. Then

\[
\text{McKay quiver of } G = \mathbb{Z}/2\text{-orbifold of the McKay quiver of } A
\]

Fixed points \( \rho_j \in \text{Irr } A \) by \( \beta \) become two 1-dimensional representations \( \rho_j^+ \) and \( \rho_j^- \). Free orbits \( \{ \rho_i^+, \rho_i^- \} \) by \( \beta \) become one 2-dimensional representation \( V_r \).

**Explicit description of a open cover of \( Y \)**

Let \( (Q, R) \) the bound McKay quiver, \( d = (\text{dim } \rho_i)_{i \in Q_0} \) the dimension vector and the generic stability condition \( \theta = (-\sum_{i \in Q_0} C_i \text{dim } \rho_i, 1, \ldots, 1) \). This choice of \( \theta \) implies that there exist \( \text{dim } \rho_i \) nonzero paths from the distinguished source \( \rho_0^+ \) to every other irreducible representation \( \rho_j \).

Any G-graph \( \Gamma \) produces the choices for nonzero maps in the representation space of \((Q, R)\). Therefore, given any G-graph \( \Gamma \) we can associate an open set \( M_\Gamma \subset \mathcal{M}_\theta(Q, R) \), and the \( \{ M_\Gamma \}_{\Gamma} \) covers \( \mathcal{M}_\theta(Q, R) \).

Using the relations \( R \) of \( Q \) the equations of \( M_\Gamma \) are explicitly obtained.

**Example:** Let \( G = \langle \frac{1}{2}(1, 7), \beta \rangle \). The minimal resolution \( Y \) consists of 5 open sets given by the G-graphs \( \Gamma_2, \ldots, \Gamma_5 \). For instance, for the G-graph \( \Gamma_0 \) we have:

\[
\begin{array}{c}
\Gamma_0 \\
\rho_0^+ \rho_0^- \\
G\text{-Hilb}(\mathbb{C}^2) \\
A\text{-Hilb}(\mathbb{C}^2)/\langle \beta \rangle
\end{array}
\]

and \( M_{\Gamma_1} \) is given by \( (cd = (1 + ca^2)E) \subset \mathbb{C}^3 \)

Remaining open sets for \( \mathcal{M}_\theta(Q, R) \) as hypersurfaces in \( \mathbb{C}^3 \):

- \( M_{\Gamma_2} : b_2^x E = (b_2^x + 1)D^+ \)
- \( M_{\Gamma_3} : b_3^2 G = (b_3^2 + 1)D^- \)
- \( M_{\Gamma_4} : \varepsilon f = (\varepsilon f - 1)D_x \)
- \( M_{\Gamma_5} : gh = (g^2 h - 1)D_- \)