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QUANTUM RULED SURFACES DEFINED BY QUIVERS

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Abstract. There have been two major achievements in noncommutative algebraic geometry, namely, the classification of noncommutative projective curves, and the classification of quantum projective planes. In both classifications, geometric triples played an essential role. In this paper, we interpret Van den Bergh's definition of a quantum ruled surface in terms of a quiver, which is nowadays one of the main tools in representation theory of finite dimensional algebras, and classify "decomposable" quantum ruled surfaces using geometric triples.

1. Quasi-schemes

Throughout, we fix an algebraically closed field k. By [8] and [16], every scheme X can be reconstructed from the category $\operatorname{Qcoh} X$ of quasi-coherent sheaves on X, so we will extend the notion of scheme as follows.

Definition. [15], [20] A **quasi-scheme** X is a Grothendieck category $\operatorname{Mod} X$. We say that a quasi-scheme X is noetherian if $\operatorname{Mod} X$ is locally noetherian, that is, $\operatorname{Mod} X$ has a small set of noetherian generators. In this case, we denote by $\operatorname{mod} X \subset \operatorname{Mod} X$ the full subcategory consisting of noetherian objects.

By [18], $\operatorname{Qcoh} X$ is a Grothendieck category for a usual scheme X, so we will view a scheme X as a quasi-scheme by $\operatorname{Mod} X = \operatorname{Qcoh} X$. The above notion of quasi-scheme includes noncommutative schemes. For example, the **noncommutative affine scheme** $X = \operatorname{Spec} R$ associated to a ring R is defined to be a quasi-scheme where $\operatorname{Mod} X = \operatorname{Mod} R$ is the category of right R-modules. Our main object of study is a noncommutative projective scheme defined as follows. Let A be a graded ring. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules, and by $\operatorname{Tors} A \subset \operatorname{GrMod} A$ the full subcategory consisting of direct limits of right bounded modules (i.e. $M_n = 0$ for all $n \gg 0$).

Definition. [6] The **noncommutative projective scheme** $X = \operatorname{Proj} A$ associated to a graded ring A is a quasi-scheme where

$$\operatorname{Mod} X = \operatorname{Tails} A := \operatorname{GrMod} A / \operatorname{Tors} A$$

is the quotient category.

The above definition can be justified by the following classical theorem.

This is an expository paper. The detailed version of this paper will be submitted for publication elsewhere.

Theorem. [17] If A is a commutative graded algebra finitely generated in degree 1 over k and $X = \operatorname{Proj} A$ in the usual sense, then

$$\operatorname{Tails} A \cong \operatorname{Mod} X := \operatorname{Qcoh} X.$$

A **noncommutative projective variety of dimension** d is a quasi-scheme $\operatorname{Proj} A$ for some graded domain A finitely generated in degree 1 over k such that $\operatorname{GKdim} A = d + 1$ where $\operatorname{GKdim} A$ is the Gelfand-Kirillov dimension of A. One of the major projects in noncommutative algebraic geometry is to classify noncommutative projective varieties of low dimensions.

2. Some Classification Results

There have been two major achievements in noncommutative algebraic geometry, namely, the classification of noncommutative projective curves, and the classification of quantum projective planes. A geometric triple defined below plays an essential role in these classifications.

Definition. [4], [5] A **geometric triple** $(X, \sigma, \mathcal{L})$ consists of a scheme X, an automorphism $\sigma \in \operatorname{Aut}_k X$, and an invertible sheaf $\mathcal{L} \in \operatorname{Pic} X$. For such a triple, we define two graded algebras as follows:
(**A-construction**)

$$A(X, \sigma, \mathcal{L}) := \frac{T(V)}{(\{f \in V \otimes V \mid f(\Delta_\sigma) = 0\})}$$

where $V = \mathrm{H}^0(X, \mathcal{L})$, and $\Delta_\sigma = \{(p, \sigma(p)) \mid p \in X\} \subset X \times X$ is the graph of σ.
(**B-construction**)

$$B(X, \sigma, \mathcal{L}) := \bigoplus_{i \in \mathbb{N}} \mathrm{H}^0(X, \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \cdots \otimes (\sigma^{i-1})^* \mathcal{L})$$

where the product of $a \in B(X, \sigma, \mathcal{L})_i$ and $b \in B(X, \sigma, \mathcal{L})_j$ is defined by

$$ab := a \otimes (b \circ \sigma^i).$$

There is a notion of ampleness for $\mathcal{L}$ with respect to σ, having an expected property as follows.

Theorem. [4] Let $(X, \sigma, \mathcal{L})$ be a geometric triple. If $\mathcal{L}$ is σ-ample, then

$$\operatorname{Tails} B(X, \sigma, \mathcal{L}) \cong \operatorname{Mod} X := \operatorname{Qcoh} X.$$

Noncommutative projective curves were classified by Artin-Stafford (1995) as follows:

Theorem. [3] For a noncommutative projective curve Y, there exists a geometric triple $(X, \sigma, \mathcal{L})$ where X is a commutative curve and $\mathcal{L}$ is σ-ample such that

$$\operatorname{Mod} Y \cong \operatorname{Tails} B(X, \sigma, \mathcal{L}) \cong \operatorname{Mod} X.$$

that is, $Y \cong X$.

The above theorem says that every noncommutative projective curve is isomorphic to a commutative curve, so the classification reduces to the commutative case. Therefore the current main project in noncommutative algebraic geometry is to classify noncommutative projective surfaces. Mimicking commutative algebraic geometry, it is reasonable to consider birational classification.

Definition. If A is a graded Ore domain, then the **function field** of $X = \operatorname{Proj} A$ is defined by $k(X) := Q_{\mathrm{gr}}(A)_0 = \{ab^{-1} \mid a, b \in A_i \text{ for some } i \in \mathbb{N}, b \neq 0\}$.

One of the motivations of noncommutative algebraic geometry is the following conjecture by Artin.

Conjecture. [1] Every noncommutative projective surface is birationally equivalent to one of the following:

(1) a quantum projective plane.
(2) a quantum ruled surface.
(3) a surface finite over its center.

Although the above conjecture is still open, it is interesting to study and classify each class of noncommutative projective surfaces above. In this paper, we will discuss the first two classes of noncommutative surfaces.

Definition. [2] A connected graded algebra A is called d-dimensional **AS-regular** if

(1) $\operatorname{gldim} A = d < \infty$,
(2) $\operatorname{GKdim} A < \infty$, and
(3) $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

A **quantum projective plane** is a noncommutative projective scheme $\operatorname{Proj} A$ for some 3-dimensional quadratic AS-regular algebra A.

The definition above can be justified by the fact that commutative AS-regular algebras are exactly (commutative) polynomial algebras. By the above definition, in order to classify quantum projective planes, it is enough to classify 3-dimensional (quadratic) AS-regular algebras. In this sense, quantum projective planes were classified by Artin-Tate-Van den Bergh (1990) as follows:

Theorem. [5] For a 3-dimensional quadratic AS-regular algebra A, there exists a geometric triple $(X, \sigma, \mathcal{L})$ where $X = \mathbb{P}^2$, or X is a cubic divisor in $\mathbb{P}^2$ such that $A(X, \sigma, \mathcal{L}) \cong A$.

Since Artin-Tate-Van den Bergh found necessary and sufficient conditions on a geometric triple $(X, \sigma, \mathcal{L})$ for $A(X, \sigma, \mathcal{L})$ to be a 3-dimensional AS-regular algebra, classification of quantum projective planes is regarded as settled, so our next project is to classify quantum ruled surfaces.

3. The First Definition of a Quantum Ruled Surface

For the rest of this paper, we assume that X is a smooth projective curve over k. First, we will review one of the characterizations of a (commutative) ruled surface. A ruled surface over X can be characterized as a scheme $\mathbb{P}_X(\mathcal{E}) := \operatorname{Proj} S(\mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank 2, and $S(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_X$. Note that $S(\mathcal{E}) = T(\mathcal{E})/(\mathcal{Q})$ where $T(\mathcal{E})$ is the tensor algebra of $\mathcal{E}$ over $\mathcal{O}_X$, and $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ is the invertible submodule locally generated by the sections of the form $x \otimes y - y \otimes x$. We want to extend this characterization of a ruled surface to a noncommutative setting. Since we have already known that every noncommutative projective curve is isomorphic to a commutative curve, it is enough to define a quantum ruled surface over a commutative curve. What we will replace is a locally free module $\mathcal{E}$ by a locally free bimodule. If R is a commutative ring, then R-bimodules are the same as $R \otimes R$-modules. If $X = \operatorname{Spec} R$, then $\operatorname{Spec}(R \otimes R) = X \times X$, so we may think of X-bimodules as $X \times X$-modules.

Definition. [4] A coherent $\mathcal{O}_X$**-bimodule** is a coherent sheaf $\mathcal{M}$ on $X \times X$ such that

$$pr_i : \operatorname{Supp} \mathcal{M} \subset X \times X \to X$$

are finite maps where $pr_i(p_1, p_2) = p_i$ are the restrictions of the projection maps.

A coherent $\mathcal{O}_X$-bimodule $\mathcal{E}$ is called **locally free of rank** r if $pr_{i*}\mathcal{E}$ are locally free of rank r on X for $i = 1, 2$.

If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-bimodule, then

$$- \otimes_X \mathcal{M} : \operatorname{Mod} X \longleftrightarrow \operatorname{Mod} X : \mathcal{H}om_X(\mathcal{M}, -)$$

is an adjoint pair of functors where

$$- \otimes_X \mathcal{M} := pr_{2*}(pr_1^*(-) \otimes_{X \times X} \mathcal{M})$$
$$\mathcal{H}om_X(\mathcal{M}, -) := pr_{1*}(\mathcal{H}om_{X \times X}(\mathcal{M}, pr_2^!(-))).$$

The following is the key lemma which makes it possible to define a quantum ruled surface.

Lemma 3.1. [21] If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank r, then there exist locally free $\mathcal{O}_X$-bimodules $\mathcal{E}^*$ and ${}^*\mathcal{E}$ of rank r such that $- \otimes_X \mathcal{E}^* \cong \mathcal{H}om_X(\mathcal{E}, -)$ is a right adjoint to $- \otimes_X \mathcal{E}$ and $- \otimes_X {}^*\mathcal{E}$ is a left adjoint to $- \otimes_X \mathcal{E}$.

Definition. [21] An invertible $\mathcal{O}_X$-subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ is called **non-degenerate** if the composition

$$\mathcal{E}^* \otimes_X \mathcal{Q} \to \mathcal{E}^* \otimes_X \mathcal{E} \otimes_X \mathcal{E} \to \mathcal{E}$$

is an isomorphism where the first map is induced by the inclusion $\mathcal{Q} \to \mathcal{E} \otimes_X \mathcal{E}$ and the second map is induced by the adjoint map $\mathcal{E}^* \otimes_X \mathcal{E} \to \mathcal{O}_X$.

The below is the first definition of a quantum ruled surface.

Definition. [19], [21] A **quantum ruled surface** over X is a quasi-scheme $\mathbb{P}_X(\mathcal{E}) := \operatorname{Proj} \mathcal{A}$ where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2, $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ is a non-degenerate invertible $\mathcal{O}_X$-subbimodule, and $\mathcal{A} = T(\mathcal{E})/(\mathcal{Q})$.

By the above definition, the Grothendieck group of a quantum ruled surface can be computed explicitly. Such an explicit calculation will be useful in applying intersection theory.

Theorem. [13] If $\mathbb{P}_X(\mathcal{E})$ is a quantum ruled surface over X, then

$$K_0(\mathbb{P}_X(\mathcal{E})) := K_0(\operatorname{mod} \mathbb{P}_X(\mathcal{E})) \cong \frac{K_0(X)[t]}{([\mathcal{O}_X] - [pr_{2*}\mathcal{E}]t + [pr_{2*}\mathcal{Q}]t^2)}.$$

Unfortunately, the above definition of a quantum ruled surface has some disadvantages. For example, given a locally free $\mathcal{O}_X$-bimodule $\mathcal{E}$, it is not clear if a non-degenerate invertible subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ exists. Even if it exists, it is not clear if $\mathbb{P}_X(\mathcal{E})$ is independent of the choice of $\mathcal{Q}$. In order to avoid these problems, we will redefine a quantum ruled surface in terms of a quiver, which is nowadays one of the mail tools in representation theory of finite dimensional algebras.

4. Quantum Ruled Surfaces Defined by Quivers

A quiver $Q = (Q_0, Q_1, h, t)$ consists of a set of vertices Q_0, a set of arrows Q_1, and two maps $h, t : Q_1 \to Q_0$. A path of length n is a sequence of arrows $x_1 x_2 \cdots x_n$ where $h(x_i) = t(x_{i+1})$ for all $i = 1, \dots, n-1$. Each vertex i can be regarded as a path e_i of length 0. The **path algebra** ΛQ of a quiver Q is a vector space spanned by all paths with the multiplication defined by the concatenation of paths. For a quiver Q, we define the double of Q by

$$\begin{aligned} \overline{Q} &= (Q_0, \{x, x^* \mid x \in Q_1\}, \overline{h}, \overline{t}) \\ \overline{h}(x) &:= h(x) =: \overline{t}(x^*) \\ \overline{t}(x) &:= t(x) =: \overline{h}(x^*). \end{aligned}$$

The **preprojective algebra** ΠQ of a quiver Q is the path algebra of the quiver $\overline{Q}$ modulo the ideal generated by

$$\sum_{x \in Q_1} (xx^* - x^*x).$$

Fortunately, in this paper, we need to understand only one example below.

Example. If

$$Q = \bullet \;\underset{y}{\overset{x}{\rightrightarrows}}\; \bullet$$

is a quiver, then the path algebra is $R = \Lambda Q \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$. Moreover,

$$\overline{Q} = \bullet \;\begin{smallmatrix} \xrightarrow{x} \\ \xrightarrow{y} \\ \xleftarrow{x^*} \\ \xleftarrow{y^*} \end{smallmatrix}\; \bullet$$

is the double of Q, and the preprojective algebra is

$$A = \Pi Q = \Lambda\overline{Q}/(xx^* + yy^*, x^*x + y^*y).$$

In this case, it is classical that $\operatorname{Tails} A \cong \operatorname{Mod} \mathbb{P}^1$ that is, $\operatorname{Proj} A \cong \mathbb{P}^1$, and $\mathcal{D}^b(\operatorname{mod} R) \cong \mathcal{D}^b(\operatorname{mod} \mathbb{P}^1)$.

A ruled surface is a $\mathbb{P}^1$-bundle over X, so, by the above example, we want to define a quantum ruled surface to be the noncommutative projective scheme associated to the preprojective algebra of the quiver $Q := X \xrightarrow{\mathcal{E}} X$. In order to do it, we need to extend the notion of quiver Q so that Q_0 is a set of k-linear categories, Q_1 is a set of k-linear functors, etc. We will explain it by an example.
Let V be a finite dimensional vector space over k. (In the above Example, $V = kx + ky$.) For a quiver

$$Q = k \xrightarrow{V} k := \operatorname{Mod} k \xrightarrow{-\otimes_k V} \operatorname{Mod} k,$$

the path algebra of Q can be defined so that $\Lambda Q \cong \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$, and the preprojective algebra of Q can be defined by $\Pi Q := \Lambda\overline{Q}/(\operatorname{Im}\varphi, \operatorname{Im}\psi)$ where $\overline{Q} = k \underset{V^*}{\overset{V}{\rightleftarrows}} k$ is the double of Q, and $\varphi : k \to V \otimes_k V^*, \psi : k \to V^* \otimes_k V^{**} \cong V^* \otimes_k V$ are the adjoint maps. If we want to extend the above construction to the quiver

$$Q = X \xrightarrow{\mathcal{E}} X := \operatorname{Mod} X \xrightarrow{-\otimes_X \mathcal{E}} \operatorname{Mod} X$$

where X is a smooth projective scheme and $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule, then there is no problem to define the path algebra Q so that $\Lambda Q \cong \begin{pmatrix} \mathcal{O}_X & \mathcal{E} \\ 0 & \mathcal{O}_X \end{pmatrix}$, however, there is a problem to define the preprojective algebra of Q by $\Pi Q := \Lambda\overline{Q}/(\operatorname{Im}\varphi, \operatorname{Im}\psi)$ where $\overline{Q} := X \underset{\mathcal{E}^*}{\overset{\mathcal{E}}{\rightleftarrows}} X$ is the double of Q, and $\varphi : \mathcal{O}_X \to \mathcal{E} \otimes_X \mathcal{E}^*, \psi : \mathcal{O}_X \to \mathcal{E}^* \otimes_X \mathcal{E}$ because there is no canonical map $\psi : \mathcal{O}_X \to \mathcal{E}^* \otimes_X \mathcal{E}$ unless $\mathcal{E} \cong \mathcal{E}^{**}$, so we will modify the definition. The definition below was originally given by Van den Bergh without using quivers.

Definition. [21], [11] Let $Q = X \xrightarrow{\mathcal{E}} X$ be a quiver where X is a smooth projective curve and $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2. The preprojective algebra $\mathcal{A} = \Pi Q$ of a quiver Q is the path algebra of the quiver

$$\overline{Q} := \cdots \xrightarrow{{}^{**}\mathcal{E}} X \xrightarrow{{}^{*}\mathcal{E}} X \xrightarrow{\mathcal{E}} X \xrightarrow{\mathcal{E}^*} X \xrightarrow{\mathcal{E}^{**}} X \xrightarrow{\mathcal{E}^{***}} \cdots$$

modulo the ideal generated by $\mathcal{Q}_i := \operatorname{Im}\varphi_i$ where $\varphi_i : \mathcal{O}_X \to \mathcal{E}^{i*} \otimes \mathcal{E}^{(i+1)*}$ are the adjoint maps. In this setting, $\mathbb{P}_X(\mathcal{E}) := \operatorname{Proj} \mathcal{A}$ is called a **quantum ruled surface**.

Note that $\mathbb{P}_X(\mathcal{E})$ is well-defined for every $\mathcal{E}$, independent of the choice of $\mathcal{Q}$, and agrees with the old definition [21].

We can define various notions for a quantum ruled surface by the above definition. Below, we denote by $\pi : \mathrm{GrMod}\,\mathcal{A} \to \mathrm{Tails}\,\mathcal{A}$ the quotient functor, which is an exact functor, and by $\omega : \mathrm{Tails}\,\mathcal{A} \to \mathrm{GrMod}\,\mathcal{A}$ the section functor, which is the right adjoint to π.

Definition. [9] For a quantum ruled surface $\mathbb{P}_X(\mathcal{E})$, we define the following.

(1) The **structure maps** $f_i : \mathbb{P}_X(\mathcal{E}) \to X$ are the adjoint pairs of functors

$$f_i^* : \mathrm{Mod}\,X \xrightarrow{-\otimes e_i\mathcal{A}} \mathrm{GrMod}\,\mathcal{A} \xrightarrow{\pi} \mathrm{Tails}\,\mathcal{A}$$

$$f_{i*} : \mathrm{Tails}\,\mathcal{A} \xrightarrow{\omega} \mathrm{GrMod}\,\mathcal{A} \xrightarrow{(-)_i} \mathrm{Mod}\,X.$$

where $\{e_i\mathcal{A}\}$ is the set of "indecomposable projective" $\mathcal{A}$-modules.

(2) The **structure sheaf** on $\mathbb{P}_X(\mathcal{E})$ is

$$\mathcal{O}_{\mathbb{P}_X(\mathcal{E})} := f_0^*\mathcal{O}_X \in \mathrm{mod}\,\mathbb{P}_X(\mathcal{E}).$$

(3) The **canonical sheaf** on $\mathbb{P}_X(\mathcal{E})$ is

$$\omega_{\mathbb{P}_X(\mathcal{E})} := f_2^*(\omega_X \otimes_X \mathcal{Q}_0) \in \mathrm{mod}\,\mathbb{P}_X(\mathcal{E}),$$

where ω_X is the canonical sheaf on X.

The canonical sheaf behaves as we expect.

Theorem. [9] (Serre Duality) If $\mathbb{P}_X(\mathcal{E})$ is a quantum ruled surface, then

$$\mathrm{Ext}^i_{\mathbb{P}_X(\mathcal{E})}(\mathcal{M}, \omega_{\mathbb{P}_X(\mathcal{E})}) \cong \mathrm{Ext}^{2-i}_{\mathbb{P}_X(\mathcal{E})}(\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}, \mathcal{M})^*$$

for all $\mathcal{M} \in \mathrm{mod}\,\mathbb{P}_X(\mathcal{E})$ where $(-)^*$ is the functor taking the k-vector space dual.

5. Geometry of Quantum Ruled Surfaces

Since a quasi-scheme is a category, there is no geometry in a genuine sense, however, by extending intersection theory to a noncommutative setting, we can see the geometry of curves on a quantum ruled surface.

Definition. Let Y be a noetherian quasi-scheme over k.

(1) We say that Y is Ext-**finite** if $\dim_k \mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{N}) < \infty$ for all $i \in \mathbb{N}$, and all $\mathcal{M}, \mathcal{N} \in \mathrm{mod}\,Y$.

(2) The **homological dimension** of Y is

$$\mathrm{hd}(Y) := \sup\{i \mid \mathrm{Ext}^i_Y(-,-) \neq 0\}.$$

If Y is a noetherian Ext-finite quasi-scheme over k of finite homological dimension, then the Euler form can be extended to the Grothendieck group as

$$\xi(-,-) : K_0(Y) \times K_0(Y) \to \mathbb{Z}$$

$$\xi([\mathcal{M}],[\mathcal{N}]) := \sum_{i\in\mathbb{N}} (-1)^i \dim_k \mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{N}).$$

Definition. [12] The **intersection multiplicity** of $[\mathcal{M}], [\mathcal{N}] \in K_0(Y)$ is defined by

$$[\mathcal{M}] \cdot [\mathcal{N}] := (-1)^{\mathrm{codim}\,\mathcal{M}} \xi([\mathcal{M}], [\mathcal{N}]).$$

It is not clear if the above definition of intersection multiplicity agrees with the usual one for commutative schemes. Fortunately, this is the case by the theorem below.

Theorem. [7] If Y is a smooth projective variety over k, and C, D are subvarieties of Y such that

$$\dim C + \dim D \leq \dim Y,$$

then

$$C \cdot D = \mathcal{O}_C \cdot \mathcal{O}_D$$

where the left hand side is the usual intersection multiplicity defined by Tor and the right hand side is our intersection multiplicity defined by Ext.

It was proved that a quantum ruled surface is Ext-finite [14] and has finite homological dimension [9], so we can apply our intersection theory to a quantum ruled surface. It is unclear what should be curves on a noncommutative surface in general, however, the following are reasonable to be called curves (divisors) on a quantum ruled surface. Here the Grothendieck group plays an essential role.

Definition. [13], [9] For a quantum ruled surface $\mathbb{P}_X(\mathcal{E})$, we define the following.

(1) The **fiber** $f^{-1}p$ of a closed point $p \in X$ is

$$\mathcal{O}_{f^{-1}p} := [f_0^* \mathcal{O}_p] \in K_0(\mathbb{P}_X(\mathcal{E})).$$

(2) The **section** H on $\mathbb{P}_X(\mathcal{E})$ is

$$\mathcal{O}_H := [f_0^* \mathcal{O}_X] - [f_1^* \mathcal{O}_X] \in K_0(\mathbb{P}_X(\mathcal{E})).$$

(3) The **canonical divisor** K on $\mathbb{P}_X(\mathcal{E})$ is

$$\mathcal{O}_K := [\omega_{\mathbb{P}_X(\mathcal{E})}] - [\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}] \in K_0(\mathbb{P}_X(\mathcal{E})).$$

If C and D are divisors as above, then we define

$$C \cdot D := \mathcal{O}_C \cdot \mathcal{O}_D = -\xi(\mathcal{O}_C, \mathcal{O}_D).$$

Below, we list the results about intersection theory on a quantum ruled surface. The first result completely determines the intersection theory on $\operatorname{Pic} \mathbb{P}(\mathcal{E})$. Roughly speaking, two distinct fibers do not meet, and a fiber and the section meet exactly once. It justifies that geometry of a quantum ruled surface behaves like that of a commutative ruled surface.

Theorem. [13], [9] Let $\mathbb{P}_X(\mathcal{E})$ be a quantum ruled surface.

(1) For closed points $p, q \in X$ and the section H on $\mathbb{P}_X(\mathcal{E})$,

$$\begin{aligned} f^{-1}p \cdot f^{-1}q &= 0 \\ f^{-1}p \cdot H &= 1 \\ H \cdot f^{-1}q &= 1 \\ H \cdot H &= \deg(pr_{2*}\mathcal{E}). \end{aligned}$$

(2) The canonical divisor K on $\mathbb{P}_X(\mathcal{E})$ is numerically equivalent to
$$-2H + (2g(X) - 2 - e)f^{-1}p$$
where $p \in X$ is a closed point, $g(X)$ is the genus of X, and $e := -H \cdot H$.
(3) (Adjunction Formula) If K is the canonical divisor on $\mathbb{P}_X(\mathcal{E})$, and D is either a fiber $f^{-1}p$ or the section H, then
$$2g(D) - 2 = D \cdot D + D \cdot K,$$
where $g(D) = 1 - \chi(\mathcal{O}_D)$ is the genus of D.

6. Classification

In classifying quantum ruled surfaces, it is important to know necessary and/or sufficient conditions on locally free $\mathcal{O}_X$-bimodules $\mathcal{E}, \mathcal{F}$ of rank 2 for $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{F})$. In the commutative case, the answer is the existence of an invertible sheaf $\mathcal{L}$ such that $\mathcal{F} \cong \mathcal{E} \otimes_X \mathcal{L}$. Using a quiver, one direction of this result can be easily extended to the noncommutative case.

Theorem. [10] If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2, and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-bimodule, then $\mathbb{P}_X(\mathcal{E} \otimes_X \mathcal{L}) \cong \mathbb{P}_X(\mathcal{E})$.

We call a quantum ruled surface Y decomposable if $Y \cong \mathbb{P}_X(\mathcal{E})$ where $\mathcal{E}$ can be written as a direct sum of two locally free bimodules of rank 1. The following corollary is immediate from the above theorem.

Corollary. [10] For a decomposable quantum ruled surface Y, there exists a geometric triple $(X, \sigma, \mathcal{L})$ such that
$$Y \cong \mathbb{P}_X(\mathcal{O}_X \oplus (pr_1^*\mathcal{L} \otimes_{X \times X} \mathcal{O}_{\Delta_\sigma}))$$
where $\Delta_\sigma = \{(p, \sigma(p)) \mid p \in X\} \subset X \times X$ is the graph of σ.

It is interesting that a geometric triple is again used in the classification.

In order to make the best use of techniques of representation theory of finite dimensional algebras, it is important to show the following.

Question. If $\mathcal{R} = \Lambda Q \cong \begin{pmatrix} \mathcal{O}_X & \mathcal{E} \\ 0 & \mathcal{O}_X \end{pmatrix}$ is the path algebra of the quiver $Q = X \xrightarrow{\mathcal{E}} X$, then $\mathcal{D}^b(\operatorname{mod} \mathbb{P}_X(\mathcal{E})) \cong \mathcal{D}^b(\operatorname{mod} \mathcal{R})$.

There are a few evidences for the above question to be true. (We state the results below without defining some of the notations.)

Lemma 6.1. [11] Let
$$\mathcal{T} := f_0^*\mathcal{O}_X \oplus f_1^*\mathcal{O}_X \in \operatorname{mod} \mathbb{P}_X(\mathcal{E}).$$

(1) $\mathcal{E}xt^i_{\mathbb{P}_X(\mathcal{E})}(\mathcal{T}, \mathcal{T}) \cong \begin{cases} \mathcal{R} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$
(2) $- \otimes_{\mathcal{R}} \mathcal{T} : \operatorname{mod} \mathcal{R} \leftrightarrow \operatorname{mod} \mathbb{P}_X(\mathcal{E}) : \mathcal{H}om_{\mathbb{P}_X(\mathcal{E})}(\mathcal{T}, -)$ is an adjoint pair of functors.

(3) $\mathcal{H}om_{\mathbb{P}_X(\mathcal{E})}(\mathcal{T}, - \otimes_{\mathcal{R}} \mathcal{T}) \cong \mathrm{Id}_{\mathrm{mod}\,\mathcal{R}}$.
(4) $\mathcal{T}$ generates $\mathcal{D}^b(\mathrm{mod}\,\mathbb{P}_X(\mathcal{E}))$ relative to X.

By the above lemma, we expect that $\mathcal{T}$ is a tilting generator for $\mathbb{P}_X(\mathcal{E})$ relative to X, inducing the expected equivalence of derived categories.

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