代数幾何学シンポジウム記録
2009年度 pp.39-53

Equivariant sheaves and their applications to invariant theory

Mitsuyasu Hashimoto

Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464–8602 JAPAN
hasimoto@math.nagoya-u.ac.jp

1. Introduction

Let S be a noetherian scheme, G a flat S-group scheme of finite type, and X a G-scheme, that is, an S-scheme with a (left) G-action. Roughly speaking, a G-equivariant $\mathcal{O}_X$-module is an $\mathcal{O}_X$-module with a G-action. It is nothing but a G-linearized $\mathcal{O}_X$-module, and this was treated by Mumford [13]. In affine settings, it is nothing but a (G, A)-module discussed in [14]. The category of (quasi-coherent) G-equivariant $\mathcal{O}_X$-modules can be embedded in the category of modules over the truncated simplicial scheme arising from the G-action on X.

The purpose of these notes is to give a survey on the study of G-equivariant sheaves utilizing general diagrams of schemes in [5]. In particular, we discuss the construction of equivariant twisted inverses, equivariant dualizing complexes, and their application to invariant theory.

We also introduce some recent applications of equivariant sheaves, based on [7], [8], and [9]. In short, they are equivariant versions of some theories in commutative algebra, plus their applications. In particular, we discuss G-local G-schemes, which is an equivariant version of local schemes (local rings), and G-local cohomology (an equivariant version of the local cohomology). Equivariant versions of Matlis and the local dualities are given.

Section 2 is a short survey on equivariant modules and their realization as sheaves over a diagram of schemes. Section 3 treats equivariant twisted inverses, equivariant dualizing complexes, and their application to invariant theory (a generalization of Watanabe's theorem on Gorenstein property of invariant subrings). Section 4 treats some recent applications of equivariant sheaves.

2. The category of $(G, \mathcal{O}_X)$-modules

Let k be an algebraically closed field and G an affine algebraic group over k.

Definition 2.1. Let A be a commutative G-algebra. We say that M is a (G, A)-module if

- M is a G-module;
- M is an A-module;
- the k-space structures of M coming from the above two items agree;
- The action $A \otimes M \to M$ ($a \otimes m \mapsto am$) is G-linear.

(G, A)-modules play important role in invariant theory. They are treated in [14] for example. The following theorem was proved using homological algebra of (G, A)-modules.

Let k be an algebraically closed field of characteristic $p > 0$. Let G be a reductive group over k. Let U be the unipotent radical of a Borel subgroup of G. Let V be a finite dimensional G-module. Let $C := \operatorname{Sym} V$ be the symmetric algebra.

Theorem 2.2 ([3], [6]). *Assume that C has a good filtration as a G-module (see [11] for the definition of good filtrations). Then*

1. *C^G is strongly F-regular. In particular, C^G is Cohen–Macaulay.*
2. *C^U is (finitely generated and) F-pure. In particular,* $\operatorname{Proj} C^U$ *is Frobenius split.*

The notion of (G, A)-modules is generalized to that of G-linearized $\mathcal{O}_X$-modules. It is convenient to introduce them as equivariant sheaves over diagrams of schemes.

Let S be the base scheme, and let $\underline{\mathrm{Sch}}/S$ be the category of S-schemes. Let I be a small category. Let $X_\bullet$ be an I^{op}-diagram of S-schemes. That is, let $X_\bullet \in \operatorname{Func}(I^{\mathrm{op}}, \underline{\mathrm{Sch}}/S)$ be a contravariant functor form I to $\underline{\mathrm{Sch}}/S$.

Definition 2.3. We define a category $\mathrm{Zar}(X_\bullet)$ by:

$$\mathrm{ob}(\mathrm{Zar}(X_\bullet)) := \{(i, U) \mid i \in \mathrm{ob}\, I,\ U \in \mathrm{ob}(\mathrm{Zar}\, X_i)\};$$

$$\mathrm{Zar}(X_\bullet)((j, V), (i, U)) := \{(\phi, h) \mid \phi \in I(i, j),$$
$$h : V \to U,\ \ \begin{array}{ccc} V & \xrightarrow{h} & U \\ \downarrow & & \downarrow \\ X_j & \xrightarrow{X_\phi} & X_i \end{array} \text{ is commutative}\}.$$

The composition is given by $(\phi, h) \circ (\phi', h') = (\phi'\phi, hh')$.

(2.4) We introduce a Grothendieck topology into $\mathrm{Zar}(X_\bullet)$. A class of morphisms $((i_\lambda, U_\lambda) \xrightarrow{(\phi_\lambda, h_\lambda)} (i, U))_{\lambda \in \Lambda}$ is said to be a covering if $\forall \lambda\ i_\lambda = i,\ \phi_\lambda = \mathrm{id}_i$, and $U = \bigcup_\lambda h_\lambda(U_\lambda)$.

Moreover, defining $\Gamma((i, U), \mathcal{O}_{X_\bullet}) := \Gamma(U, \mathcal{O}_{X_i})$, $\mathcal{O}_{X_\bullet}$ is a sheaf of commutative rings on $\mathrm{Zar}(X_\bullet)$. Thus $\mathrm{Zar}(X_\bullet)$ is a ringed site.

We denote the category $\mathrm{Mod}(\mathrm{Zar}(X_\bullet))$ simply by $\mathrm{Mod}(X_\bullet)$.

(2.5) Next we introduce the restriction functor.

For $i \in \mathrm{ob}(I)$, we define $(?)_i : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(X_i)$ by $\Gamma(U, \mathcal{M}_i) := \Gamma((i, U), \mathcal{M})$ for $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$. $(?)_i$ is called the restriction functor. Note that $(?)_i$ has both a left adjoint and a right adjoint. In particular, $(?)_i$ preserves arbitrary limits and colimits. In particular, $(?)_i$ is exact.

(2.6) Next we introduce the β map.

For $\phi \in I(i, j)$, we define $\beta_\phi : (?)_i \to (X_\phi)_*(?)_j$ by

$$\Gamma(U, \mathcal{M}_i) = \Gamma((i, U), \mathcal{M}) \xrightarrow{\mathrm{res}_{(\phi, X_\phi|_{X_\phi^{-1}(U)})}} \Gamma((j, X_\phi^{-1}(U)), \mathcal{M}) = \Gamma(X_\phi^{-1}(U), \mathcal{M}_j) = \Gamma(U, (X_\phi)_*\mathcal{M}_j).$$

(2.7) For $\phi \in I(i, j)$, we define $\alpha_\phi : X_\phi^*(?)_i \to (?)_j$ to be the composite

$$X_\phi^*(?)_i \xrightarrow{\beta_\phi} X_\phi^*(X_\phi)_*(?)_j \xrightarrow{\varepsilon} (?)_j,$$

where ε is the counit of adjunction of the adjoint pair $(X_\phi^*, (X_\phi)_*)$.

Definition 2.8. $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is said to be *equivariant* if $\alpha_\phi : X_\phi^* \mathcal{M}_i \to \mathcal{M}_j$ is an isomorphism for $\phi \in \mathrm{Mor}(I)$. The full subcategory of $\mathrm{Mod}(X_\bullet)$ consisting of equivariant $\mathcal{O}_{X_\bullet}$-modules is denoted by $\mathrm{EM}(X_\bullet)$.

Now we can define quasi-coherent and coherent sheaves.

Definition 2.9. $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is said to be:

1 *locally quasi-coherent* (resp. *locally coherent*) if $\mathcal{M}_i$ is quasi-coherent (resp. coherent) for any $i \in \mathrm{ob}(I)$.

2 *quasi-coherent* (resp. *coherent*) if it is locally quasi-coherent (resp. locally coherent) and equivariant.

The full subcategory of locally quasi-coherent (resp. quasi-coherent, coherent) modules in $\mathrm{Mod}(X_\bullet)$ is denoted by $\mathrm{Lqc}(X_\bullet)$ (resp. $\mathrm{Qch}(X_\bullet)$, $\mathrm{Coh}(X_\bullet)$).

(2.10) We define direct and inverse image functors.

Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathrm{Func}(I^{\mathrm{op}}, \underline{\mathrm{Sch}})$. Then a ringed continuous functor $f_\bullet^{-1} : \mathrm{Zar}(Y_\bullet) \to \mathrm{Zar}(X_\bullet)$ is defined by $f_\bullet^{-1}((i, U)) = (i, f_i^{-1}(U))$. Thus $(f_\bullet)_* : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(Y_\bullet)$ is defined by

$$\Gamma((i, U), (f_\bullet)_* \mathcal{M}) = \Gamma(f_\bullet^{-1}(i, U), \mathcal{M}).$$

$(f_\bullet)_*$ has a left adjoint $f_\bullet^*$.

Note that $f_\bullet^*$ preserves equivariance, local quasi-coherence, and quasi-coherence. Note also that $(f_\bullet)_*$ preserves local quasi-coherence if f_i is quasi-compact quasi-separated for each $i \in \mathrm{ob}(I)$. If, moreover, $Y_\phi f_j = f_i X_\phi$ is a cartesian square for each $\phi \in \mathrm{Mor}(I)$ ($f_\bullet$ is *cartesian*), then $(f_\bullet)_*$ also preserves quasi-coherence.

(2.11) Now we can define equivariant sheaves for a group action using sheaves over a diagram of schemes.

Let $[n]$ denote the totally ordered set $\{0, 1, \ldots, n\}$ for $n \geq -1$. Define (Δ^+) by $\mathrm{ob}(\Delta^+) = \{[n] \mid n \geq -1\}$ and $(\Delta^+)([m], [n]) = \{\varphi \in \mathrm{Map}([m], [n]) \mid \varphi \text{ is a monotone map}\}$. Define the subcategory Δ_M of (Δ^+) by $\mathrm{ob}(\Delta_M) = \{[0], [1], [2]\}$, and

$$\Delta_M([m], [n]) = \{\varphi \in (\Delta^+)([m], [n]) \mid \varphi \text{ is injective}\}.$$

Pictorially, Δ_M looks like

$$[2] \overset{\delta_0(2)}{\underset{\delta_2(2)}{\overset{\delta_1(2)}{\leftleftarrows\!\!\leftarrow}}} [1] \overset{\delta_0(1)}{\underset{\delta_1(1)}{\leftleftarrows}} [0] ,$$

where $\delta_i(j) : [j-1] \to [j]$ is the unique injective monotone map such that $i \notin \operatorname{Im} \delta_i(j)$.

(2.12) Let S be a scheme, G an S-group scheme, and X a G-scheme. We define $B_G^M(X) \in \operatorname{Func}(\Delta_M^{\mathrm{op}}, \underline{\mathrm{Sch}}/S)$ by

$$B_G^M(X) = \; G \times G \times X \overset{1\times a}{\underset{p_{23}}{\overset{\mu\times 1}{\rightrightarrows\!\!\rightarrow}}} G \times X \overset{a}{\underset{p_2}{\rightrightarrows}} X ,$$

where $a : G \times X \to X$ is the action, $\mu : G \times G \to G$ is the product, and p_{23} and p_2 are projections.

We denote $\operatorname{Mod}(B_G^M(X))$ by $\operatorname{Mod}(G, X)$ and call its object a $(G, \mathcal{O}_X)$-module. $\operatorname{Lqc}(B_G^M(X))$, $\operatorname{Qch}(B_G^M(X))$, and $\operatorname{Coh}(B_G^M(X))$ are denoted by $\operatorname{Lqc}(G, X)$, $\operatorname{Qch}(G, X)$, and $\operatorname{Coh}(G, X)$, respectively.

(2.13) For a G-morphism $f : X \to Y$, $B_G^M(f) : B_G^M(X) \to B_G^M(Y)$ is a cartesian morphism, and the direct image $B_G^M(f)_* : \operatorname{Mod}(G, X) \to \operatorname{Mod}(G, Y)$ and the inverse image $B_G^M(f)^* : \operatorname{Mod}(G, Y) \to \operatorname{Mod}(G, X)$ are induced.

Lemma 2.14. $\operatorname{EM}(B_G^M(X))$ *is equivalent to the category of* G*-linearized* $\mathcal{O}_X$*-modules by Mumford [13]. The equivalence induces the equivalence between* $\operatorname{Qch}(G, X)$ *and the category of quasi-coherent* G*-linearized* $\mathcal{O}_X$*-modules.*

(2.15) So we can identify an object of $\operatorname{EM}(B_G^M(X))$ by a G-linearized $\mathcal{O}_X$-module. What is the merit of considering diagrams of schemes?

- We can use induction on the number of objects of I.
- $\operatorname{Mod}(G, X) = \operatorname{Mod}(B_G^M(X))$ is a module category of a ringed site. So $\operatorname{Mod}(G, X)$ has $\underline{\mathrm{Hom}}$, $\otimes$, etc. and is flexible enough. The embedding $\operatorname{Qch}(G, X) \hookrightarrow \operatorname{Mod}(G, X)$ is a natural generalization of the embedding $\operatorname{Qch}(X) \hookrightarrow \operatorname{Mod}(X)$.
- The use of $\operatorname{Lqc}(G, X)$ is sometimes effective.

We will see that $\operatorname{Lqc}(G, X)$ plays an important role in constructing the twisted inverse functor.

(2.16) In the rest of these notes, let S be a noetherian scheme, G a flat S-group scheme of finite type, and X a noetherian G-scheme.

The following is proved using the basics on simplicial schemes, see [5, Lemma 12.8].

Lemma 2.17. *The category* $\mathrm{Qch}(G,X)$ *is a locally noetherian abelian category, and* $\mathcal{M} \in \mathrm{Qch}(G,X)$ *is a noetherian object of* $\mathrm{Qch}(G,X)$ *if and only if* $\mathcal{M} \in \mathrm{Coh}(G,X)$ *if and only if* $\mathcal{M}_{[0]}$ *is coherent as an* $\mathcal{O}_X$*-module. The forgetful functor*

$$(?)_{[0]} : \mathrm{Qch}(G,X) \to \mathrm{Qch}((B_G^M(X)_{[0]}) = \mathrm{Qch}(X)$$

given by $\mathcal{M} \mapsto \mathcal{M}_{[0]}$ *is faithful exact, and admits a right adjoint.*

(2.18) If k is a field, $S = \operatorname{Spec} k$ and G is affine, and $X = S$, then $\mathrm{Qch}(G,X)$ (resp. $\mathrm{Coh}(G,X)$) is equivalent to the category $\mathrm{Mod}(G)$ of G-modules (resp. finite dimensional G-modules). The functor

$$(?)_{[0]} : \mathrm{Qch}(G,S) \to \mathrm{Qch}(S) \cong \mathrm{Mod}(k)$$

is identified with the forgetful functor, forgetting the G-action.

Usually, a G-module and its underlying vector space are expressed by the same symbol, say V. We use this abuse of notation, and express a $(G,\mathcal{O}_X)$-module $\mathcal{M}$ and its underlying $\mathcal{O}_X$-module $\mathcal{M}_{[0]}$ by the same symbol. For example, $\mathcal{O}_{B_G^M(X)}$ is simply denoted by $\mathcal{O}_X$ because $(\mathcal{O}_{B_G^M(X)})_{[0]}$ is $\mathcal{O}_X$. For a G-morphism $f : X \to Y$, the associated direct image $B_G^M(f)_*$ is simply denoted by f_*. Similarly for $B_G^M(f)^*$.

(2.19) Let $\mathcal{M}$ be a quasi-coherent $(G,\mathcal{O}_X)$-module, and $\mathcal{N}$ a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{M}_0$. Then there is at most one $(G,\mathcal{O}_X)$-submodule $\tilde{\mathcal{N}}$ of $\mathcal{M}$ such that $\tilde{\mathcal{N}}_0 = \mathcal{N}$. In this case, we say that $\mathcal{N}$ is a $(G,\mathcal{O}_X)$-submodule of $\mathcal{M}_0$. If, moreover, $\mathcal{M} = \mathcal{O}_X$, then we say that $\mathcal{N}$ is a G-ideal of $\mathcal{O}_X$.

(2.20) Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ be in $\mathrm{Qch}(G,X)$, $\mathcal{I}$ be a G-ideal, and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_\lambda$ be quasi-coherent $(G,\mathcal{O}_X)$-submodules of $\mathcal{M}$. Let $\mathcal{L}$ and $\mathcal{M}_3$ be coherent. Then the following modules have structures of quasi-coherent $(G,\mathcal{O}_X)$-modules.

- $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$, $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{L},\mathcal{M})$,
- $\underline{H}^i_{\mathcal{I}}(\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n,\mathcal{M})$,

- The Fitting ideal $\underline{\mathrm{Fitt}}_j(\mathcal{L})$,
- $\mathcal{M}_1 \cap \mathcal{M}_2$, $\sum_\lambda \mathcal{M}_\lambda$, $\mathcal{I}\mathcal{M}_1$,
- $\mathcal{M}_1 : \mathcal{M}_3$, $\mathcal{M}_1 : \mathcal{I}$,...

3. G-dualizing complex and equivariant twisted inverse

As in the last section, until the end of these notes, let S be a noetherian scheme, G a flat S-group scheme of finite type, and X a noetherian G-scheme.

A G-dualizing complex is an equivariant analogue of a dualizing complex.

Definition 3.1. Let $\mathbb{F} \in D(G, X)$ $(= D(\mathrm{Mod}(G, X)))$. We say that $\mathbb{F}$ is *G-dualizing* if $\mathbb{F}$ has coherent cohomology groups, and the restriction $\mathbb{F}_{[0]} \in D(X)$ is a dualizing complex of X.

(3.2) If X is Gorenstein of finite Krull dimension, then $\mathcal{O}_X$ is a G-dualizing complex of X.

We say that a (non-empty noetherian) G-scheme X is G-connected if U and V are G-stable open subschemes of X, $U \cap V = \emptyset$, and $U \cup V = X$, then either $U = X$ or $V = X$ holds. It is equivalent to say that $B_G^M(X)$ is d-connected in the sense of [5].

In general, X is a disjoint union of finitely many G-stable closed open G-connected subschemes. Each of them is called a G-connected component of X.

Let X be G-connected with a fixed G-dualizing complex $\mathbb{I}$. The lowest nonzero cohomology sheaf ω_X of $\mathbb{I}$ is called the *G-canonical sheaf* of X. Note that $\omega_X \in \mathrm{Coh}(G, X)$. In general, we define ω_X G-connected componentwise.

The following is the main theorem of [5].

Theorem 1 ([5]). *Let $f : Y \to X$ be a G-morphism separated of finite type. Then there is a functor $f^! : D_{\mathrm{Lqc}}(G, X) \to D_{\mathrm{Lqc}}(G, Y)$, called the (equivariant)* twisted inverse*, which satisfies:*

- *$f^!$ is triangulated, $\mathrm{id}_X^! \cong \mathrm{Id}$, and $g^! f^! \cong (fg)^!$.*
- *$f^!(D_{\mathrm{Qch}}(G, X)) \subset D_{\mathrm{Qch}}(G, Y)$, and $f^!(D_{\mathrm{Coh}}(G, X)) \subset D_{\mathrm{Coh}}(G, Y)$.*
- *If $\mathbb{I}_X$ is G-dualizing, then $f^!(\mathbb{I}_X)$ is also G-dualizing.*

- *If f is proper, then $f^!$ is a right adjoint of $Rf_* : D_{\mathrm{Lqc}}(G,Y) \to D_{\mathrm{Lqc}}(G,X)$.*
- *If f is an open immersion, then $f^!$ agrees with the restriction f^*.*
- *If f is of finite flat dimension, then $f^!(\mathbb{F}) \cong f^!(\mathcal{O}_X) \otimes^L Lf^*\mathbb{F}$.*
- *Let $f : Y \to X$ be a finite G-morphism, and let Z denote the ringed site $(\mathrm{Zar}(B_G^M(X)), f_*\mathcal{O}_Y)$. Let $g : Z \to \mathrm{Zar}(B_G^M(Y))$ be the obvious ringed continuous functor. Then $g_\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathcal{O}_Z, ?)$ is isomorphic to $f^!$ ($g_\# : \mathrm{Mod}(Z) \to \mathrm{Mod}(G,Y)$ is the canonical functor, which is exact).*
- *If $f : Y \to X$ is a regular embedding of a well-defined codimension, say d, then $f^!\mathcal{O}_X \cong \bigwedge^d (f^*\mathcal{I})^\vee[-d]$, where $\mathcal{I}$ is the defining ideal sheaf of Y in X (it belongs to $\mathrm{Qch}(G,X)$).*
- *If $f : Y \to X$ is smooth of a well-defined relative dimension, say d, then $\Omega_{Y/X}$ has a canonical coherent $(G,\mathcal{O}_Y)$-module structure, and $f^!\mathcal{O}_X \cong \bigwedge^d \Omega_{Y/X}[d]$.*

In the construction of $f^! : D_{\mathrm{Lqc}}(G,X) \to D_{\mathrm{Lqc}}(G,Y)$, we use the existence of a factorization $B_G^M(Y) \xrightarrow{\varphi} Z \xrightarrow{\psi} B_G^M(X)$, where $Z \in \mathrm{Func}(\Delta_M^{\mathrm{op}}, \underline{\mathrm{Sch}}/S)$, ψ is (componentwise) proper, and φ is a (componentwise) image-dense open immersion. However, Z is not necessarily of the form $B_G^M(W)$ for some G-scheme W (we avoid the problem of equivariant compactification).

The following is proved using the usual duality of proper morphisms.

Theorem 2 (Duality of proper morphisms, [5, Theorem 22.5]). *Let $f : X \to Y$ be a proper G-morphism of noetherian G-schemes. Then the canonical map*

$$Rf_* R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}(G,X)}(\mathbb{F}, f^!\mathbb{G}) \to R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}(G,Y)}(Rf_*\mathbb{F}, \mathbb{G})$$

is an isomorphism for $\mathbb{F} \in D_{\mathrm{Qch}}(G,X)$ and $\mathbb{G} \in D^+_{\mathrm{Lqc}}(G,Y)$.

Corollary 3.3. *Let $S = \mathrm{Spec}\, k$, and G a reductive group over k. Let T be a maximal torus of G, and fix a base of the root system of G. Let B be the negative Borel subgroup. For any finite dimensional B-module M and any $i \in \mathbb{Z}$, there is an isomorphism of G-modules*

$$R^{n-i} \mathrm{ind}_B^G (M^* \otimes (-2\rho)) \cong (R^i \mathrm{ind}_B^G M)^*,$$

where ρ is the half sum of positive roots, and $n = \dim G/B$.

This corollary is well-known as the Serre duality for representations of reductive groups. In fact, this isomorphism is nothing but the Serre duality on G/B. See [11, (II.4.2)] for more. Usually, the Serre duality is an isomorphism of vector spaces, not an isomorphism of representations, but the theorem guarantees that it is in fact an isomorphism of representations.

We show an application of equivariant dualizing complexes and canonical sheaves to invariant theory.

Theorem 3 ([5, Proposition 32.4]). *Let k be a field, G a linearly reductive finite k-group* scheme. *Let A be a finitely generated k-algebra with a G-action. If A is Gorenstein and $\omega_A \cong A$ as a (G, A)-module, then $B := A^G$ is Gorenstein and $\omega_B \cong B$.*

As a corollary, we have

Corollary 3.4. *Let k and G be as above, and V a finite dimensional G-module. If the representation $G \to GL(V)$ factors through $SL(V)$, then $B := (\operatorname{Sym} V)^G$ is Gorenstein and $\omega_B \cong B$.*

The corollary for the case that G is a finite group is well known as a theorem of K.-i. Watanabe [15]. We give an outline of a proof of the theorem.

Proof. Cohen–Macaulay property is trivial, since $B \to A$ is finite, and B is a direct summand of A. As $\dim A = \dim B$ and $\operatorname{Ext}^i_B(A, \omega_B) = 0$ for $i > 0$,

$$\omega_A \cong \pi^! \omega_B \cong \operatorname{Hom}_B(A, \omega_B)$$

as (G, B)-modules by the equivariant duality of finite morphisms, where $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is the canonical map.

Hence

$$\omega_B \cong \operatorname{Hom}_B(B, \omega_B) \cong \operatorname{Hom}_B(A, \omega_B)^G \cong \omega_A^G \cong A^G \cong B.$$

□

4. Matijevic–Roberts type theorem and G-local G-schemes

In this section, we give various results on G-actions on schemes, based on equivariant sheaves. As in the last section, until the end of these notes, let S be a noetherian scheme, G a flat S-group scheme of finite type, and X a noetherian G-scheme.

Definition 4.1. Let Z be a closed subscheme of X. Then we denote the scheme theoretic image of the action $a : G \times Z \to X$ by Z^*.

The following hold:

- Z^* is the smallest G-stable closed subscheme of X containing Z.
- If Z is irreducible and G has connected fibers, then Z^* is irreducible.
- If Z is reduced and G is S-smooth, then Z^* is reduced.

Lemma 4.2 ([7, Corollary 6.22]). *Let Z be a G-stable closed subscheme of X. If $p_2 : G \times X \to X$ has regular fibers, then Z_{red} is G-stable. In particular, if G is S-smooth, then Z_{red} is G-stable.*

Lemma 4.3 (H—). *Let $\mathcal{I}$ be a coherent G-ideal of $\mathcal{O}_X$. If G is S-smooth, then the integral closure $\overline{\mathcal{I}}$ of $\mathcal{I}$ is again a G-ideal.*

The following is a generalized version of Matijevic–Roberts theorem. For the history of Matijevic–Roberts type theorem, see [7].

Theorem 4 ([7, Theorem 7.2]). *Assume either*

- *G is S-smooth; or*
- *$S = \operatorname{Spec} k$, where k is a perfect field.*

Let $\mathcal{C}$ and $\mathcal{D}$ be class of noetherian local rings, and assume that

- *If $A \in \mathcal{C}$ and $A \to B$ is a local homomorphism which is regular and essentially of finite type, then $B \in \mathcal{D}$; and*
- *If $B \in \mathcal{D}$ and $A \to B$ is a regular essentially of finite type local homomorphism, then $A \in \mathcal{D}$.*

Let y be a point of X, Y the closure of $\{y\}$, and let η be the generic point of an irreducible component of Y^*. If $\mathcal{O}_{X,\eta} \in \mathcal{C}$, then $\mathcal{O}_{X,y} \in \mathcal{D}$.

The following corollary is the original Matijevic–Roberts type theorem on graded rings.

Corollary 4.4. *Let A be a $\mathbb{Z}^n$-graded noetherian ring. Let P be a prime ideal of A, and let P^* be the prime ideal generated by the homogeneous elements of P.*

- *If A_{P^*} is regular, then A_P is regular.*
- *(Matijevic–Roberts [12], Hochster–Ratliff [10], Goto–Watanabe [2]) If A_{P^*} is Cohen–Macaulay (resp. Gorenstein), then A_P is Cohen–Macaulay (resp. Gorenstein).*
- *([7]) If A_{P^*} is of characteristic p, F-regular (resp. F-rational) and excellent, then A_P is F-regular (resp. F-rational).*
- *([6]) If A_{P^*} is of characteristic p and F-pure (resp. Cohen–Macaulay F-injective), then A_P is F-pure (resp. Cohen–Macaulay F-injective).*

Generalizing notions in algebraic geometry or commutative ring theory to notions in equivariant settings is an important problem. It is natural to ask, what is an equivariant version of a local ring.

Definition 4.5. A G-scheme Z is said to be G-*local* if there is a unique minimal non-empty G-stable closed subscheme P of Z. In this case, we say that (Z, P) is G-local.

If G is trivial, then a scheme Z is G-local if and only if $Z \cong \operatorname{Spec} A$ for some local ring A. For a general G, a G-local G-scheme need not be affine (see below).

Here are some examples of G-local G-schemes.

Example 4.6. Let $S = \operatorname{Spec} \mathbb{Z}$, and $G = \mathbb{G}_m^n$. Let A be a G-algebra (so A is a $\mathbb{Z}^n$-graded ring). Then $X = \operatorname{Spec} A$ is G-local if and only if A is H-local (i.e., there is a unique maximal graded ideal) as a $\mathbb{Z}^n$-graded ring.

Example 4.7. Let k be a field, and G a reductive group over k. Let A be a finitely generated G-algebra. For $\mathfrak{p} \in \operatorname{Spec} A^G$, the G-scheme $X = \operatorname{Spec} A'$ with $A' := A^G_{\mathfrak{p}} \otimes_{A^G} A$ is G-local.

Example 4.8. Let k be a field, G an affine algebraic k-group scheme, H a closed subgroup scheme of G, and $X = G/H$. Then (X, X) is G-local. So a G-local G-scheme need not be affine, even if G is so.

Example 4.9. Let k be an algebraically closed field, G a reductive group over k, and B a Borel subgroup. Then $(G/B, B/B)$ is B-local, since B/B is the smallest Schubert variety.

The following was proved using Fogarty's idea [1]. For the definition of geometric quotients, see [13].

Theorem 5 ([4]). *Let the G-scheme X be of finite type over S. If $\varphi : X \to Y$ is a universally submersive geometric quotient, then Y is of finite type over S. If $\mathcal{M}$ is a coherent $(G, \mathcal{O}_X)$-module, then $(\varphi_* \mathcal{M})^G$ is a coherent $\mathcal{O}_Y$-module.*

Here is an application of G-local G-schemes to invariant theory.

Theorem 6 ([9]). *Let k be a field, and G a linearly reductive algebraic group over k. Let Z be a noetherian, Cohen-Macaulay G-scheme with an affine geometric quotient $p : Z \to W$. Then W is (noetherian and) Cohen-Macaulay.*

In the proof, we may assume that Z is G-local. The case that G is a finite group is due to Hochster–Eagon.

(4.10) Equivariant versions of some theorems in local ring theory are obtained. Until the end of this talk, let (X, Y) be G-local, and let η be the generic point of an irreducible component of Y. Let $i : Y \hookrightarrow X$ be the inclusion.

Lemma 4.11. *The stalk functor $(?)_\eta : \mathrm{Qch}(G, X) \to \mathrm{Mod}(\mathcal{O}_{X,\eta})$ is faithful and exact.*

As a corollary, we have an equivariant version of Nakayama's lemma, which is well-known for affine case.

Lemma 4.12 (G-Nakayama's lemma). *For $\mathcal{M} \in \mathrm{Coh}(G, X)$, if $i^*\mathcal{M} = 0$, then $\mathcal{M} = 0$.*

(4.13) Next we define an equivariant version of local cohomology.

The functor $\underline{\Gamma}_Y = \mathrm{Ker}(\mathrm{Id} \to g_* g^*)$ is a functor from $\mathrm{Mod}(G, X)$ to itself, and preserves $\mathrm{Lqc}(G, X)$ and $\mathrm{Qch}(G, X)$, where $g : X \setminus Y \hookrightarrow X$ is the inclusion. The derived functor $R\underline{\Gamma}_Y : D(G, X) \to D(G, X)$ preserves $D^+_{\mathrm{Qch}}(G, X)$.

From now on, assume that X has a (fixed) G-dualizing complex $\mathbb{I}$.

Lemma 4.14. *The cohomology group of $R\underline{\Gamma}_Y(\mathbb{I})$ is concentrated in one place.*

If $\underline{H}^0_Y(\mathbb{I}) := R^0\Gamma_Y(\mathbb{I}) \neq 0$, then we say that $\mathbb{I}$ is G-normalized. From now on, we assume that $\mathbb{I}$ is G-normalized.

Definition 4.15 ([9]). We set $\mathcal{E}_X := \underline{H}^0_Y(\mathbb{I})$, and call it the *$G$-sheaf of Matlis.*

Lemma 4.16. *The stalk $\mathcal{E}_{X,\eta}$ is the injective hull of the residue field of the local ring $\mathcal{O}_{X,\eta}$.*

Thus $\mathcal{E}_X$ is the equivariant version of the injective hull of the residue field of a local ring.

The next lemma gives a characterization of quasi-coherent $(G, \mathcal{O}_X)$-modules of finite length on a G-local G-scheme X.

Lemma 4.17. *For $\mathcal{M} \in \mathrm{Qch}(G, X)$, the following are equivalent.*

- *$\mathcal{M}$ is of finite length in $\mathrm{Qch}(G, X)$.*
- *$\mathcal{M}$ is coherent in $\mathrm{Qch}(X)$, and there exists some $n \geq 0$ such that $\mathcal{I}^n\mathcal{M} = 0$, where $\mathcal{I}$ is the defining ideal of Y.*
- *$\mathcal{M}_\eta$ is an $\mathcal{O}_{X,\eta}$-module of finite length.*

Let $\mathcal{F}$ denote the full subcategory of $\mathrm{Qch}(G, X)$ consisting of objects of finite length. The following is an equivariant version of the Matlis duality.

Theorem 7 ([9]). *Let $\mathbb{D}$ denote the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{E}_X) : \mathrm{Mod}(G, X) \to \mathrm{Mod}(G, X)$.*

- *$\mathbb{D}$ is an exact functor on $\mathrm{Coh}(G, X)$.*
- *$\mathbb{D}(\mathcal{F}) \subset \mathcal{F}$.*
- *The canonical map $\mathcal{M} \to \mathbb{D}\mathbb{D}\mathcal{M}$ is an isomorphism for $\mathcal{M} \in \mathcal{F}$.*
- *$\mathbb{D} : \mathcal{F} \to \mathcal{F}$ is an anti-equivalence.*

Finally, we state the equivariant version of the local duality.

Theorem 8 (Equivariant local duality, [9]). *Let $\mathbb{F}$ be a bounded below complex in $\mathrm{Mod}(G, X)$ with coherent cohomology groups. Then there is an isomorphism in $\mathrm{Qch}(G, X)$*

$$\underline{H}^i_Y(\mathbb{F}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}), \mathcal{E}_X).$$

References

[1] J. Fogarty, Geometric quotients are algebraic schemes, *Adv. Math.* **48** (1983), 166–171.

[2] S. Goto and K.-i. Watanabe, On graded rings, II ($\mathbb{Z}^n$-graded rings), *Tokyo J. Math.* **1** (1978), 237–261.

[3] M. Hashimoto, Good filtrations of symmetric algebras and strong F-regularity of invariant subrings, *Math. Z.* **236** (2001), 605–623.

[4] M. Hashimoto, "Geometric quotients are algebraic schemes" based on Fogarty's idea, *J. Math. Kyoto Univ.* **43** (2003), 807–814.

[5] M. Hashimoto, Equivariant Twisted Inverses, *Foundations of Grothendieck Duality for Diagrams of Schemes* (J. Lipman, M. Hashimoto), Lecture Notes in Math. **1960**, Springer (2009), pp. 261–478.

[6] M. Hashimoto, Good filtrations and F-purity of invariant subrings, preprint `arXiv:0910.0081v1`

[7] M. Hashimoto and M. Miyazaki, G-prime and G-primary G-ideals on G-schemes, preprint `arXiv:0906.1441v2`

[8] M. Hashimoto and M. Ohtani, Local cohomology on diagrams of schemes, *Michigan Math. J.* **57** (2008), 383–425.

[9] M. Hashimoto and M. Ohtani, Equivariant Matlis and the local duality, preprint `arXiv:0906.3628v2`

[10] M. Hochster and L. J. Ratliff, Jr., Five theorems on Macaulay rings, *Pacific J. Math.* **44** (1973), 147–172.

[11] J. C. Jantzen, *Representations of Algebraic Groups,* 2nd ed., AMS (2003).

[12] J. Matijevic and P. Roberts, A conjecture of Nagata on graded Cohen–Macaulay rings, *J. Math. Kyoto Univ.* **14** (1974), 125–128.

[13] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory,* third edition, Springer (1994).

[14] C. S. Seshadri, Geometric reductivity over arbitrary base, *Adv. Math.* **26** (1977), 225–274.

[15] K.-i. Watanabe, Certain invariant subrings are Gorenstein I, *Osaka J. Math.* **11** (1974), 1–8.