MULTIPLIER IDEAL SHEAVES AND INTEGRAL INVARIANTS

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1. Introduction

This talk is based on joint a work with Yuji Sano.

Let M be a compact complex manifold with $c_1(M) > 0$, i.e. a Fano manifold, with $\dim M = m$.

The rst Chern class $c_1(M)$ is represented as a de Rham class by a closed positive $(1,1)$-form

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{m} g_{i\bar{j}}\ dz^i \wedge d\bar{z}^j,$$

with $(g_{i\bar{j}})$ a positive de nite Hermitian matrix.

It is well known, or by de nition, that

$$d\omega = 0 \Longleftrightarrow \omega \ \text{ is a Kähler form.}$$

We regard $c_1(M)$ as a Kähler class (the space of Kähler forms).

On the other hand, by the theory of characteristic classes (Chern-Weil Theory), $c_1(M)$ is represented by a **Ricci form**

$$\mathrm{Ric}_\omega := -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

and its coe cient

$$R_{i\bar{j}} := -\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log\det(g_{i\bar{j}})$$

is called the **Ricci curvature**.

DEF : ω is called a **Kahler-Einstein metric** if

$$\mathrm{Ric}_\omega = \omega$$

or equivalently

$$R_{i\bar{j}} = g_{i\bar{j}}.$$

But in general $\mathrm{Ric}_\omega \neq \omega$, and we have for some smooth function h

$$\mathrm{Ric}_\omega = \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}h.$$

Problem : Find another $\widetilde{\omega}$ such that

$$\mathrm{Ric}_{\widetilde{\omega}} = \widetilde{\omega}.$$

If we put

$$\widetilde{\omega} = \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi,$$

the Einstein equation

$$\mathrm{Ric}_{\widetilde{\omega}} = \widetilde{\omega}$$

is equivalent to the complex Monge-Ampère equation

$$\frac{\det(g_{ij} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j})}{\det(g_{ij})} = e^{\ \varphi+h}.$$

Thus, starting from arbitrary $\omega \in c_1(M)$, nding a Kähler-Einstein metric with $\tilde{\omega} \in c_1(M)$ is reduced to solving the non-linear PDE

$$\frac{\det(g_{ij} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j})}{\det(g_{ij})} = e^{\ \varphi+h}.$$

Conjecture (Yau-Tian-Donaldson) :
The existence of a Kähler-Einstein metric will be
equivalent to GIT stability (K-polystability).

2. Obstructions

On the one hand there are **obstructions** to $\exists$ of K-E metrics by Matsushima, the speaker, Bando-Mabuchi, Chen-Tian-Donaldson-Stoppa-Mabuchi, ... as below.

Matsushima (1956) : If M admits a K-E metric then the Lie algebra $\mathfrak{h}(M)$ of all holomorphic vector elds is reductive.

Futaki (1983) : $\exists$ Lie algebra character $f : \mathfrak{h}(M) \to \mathbb{C}$ such that if $\exists$ K-E metric then $f = 0$. This f is called the so-called "Futaki invariant", and the precise de nition will be given below.

Bando-Mabuchi (1987) : K-energy is bounded from below.

Chen-Tian, Donaldson, Stoppa, Mabuchi, ... : Existence of K-E $\Longrightarrow$ K-stability.

The de nition of K-stability is roughly stated as follows.

Definition
M is K-stable. $\Longleftrightarrow$
For all C -equivariant degenarations (test con gurations) of M, the central ber has positive Donaldson's Futaki invariant. (The minus of the Futaki invariant is the invariant used as the analogy to the numerical criterion of GIT.)

M is K-polystable. $\Longleftrightarrow$
For all C -equivariant degenarations (test con gurations) of M, the central ber has non-negative Futaki invariant, and the equality occurs only when the test con guration is a product $M \times \mathbb{C}$ with non-trivial $\mathbb{C}$ -action on M. (In this case Futaki invariant necessarily vanishes because we may also consider the opposite $\mathbb{C}$ -action.)

Definition of f : Recall that

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{ij}\ dz^i \wedge d\overline{z}^j,$$

$$\rho_\omega = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det(g_{ij}),$$

and

$$\rho_\omega - \omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial}h, \qquad h \in C^\infty(M).$$

Then f is de ned by

$$f(X) = \int_M Xh\ \omega^m$$

for $X \in \mathfrak{h}(M)$.

Theorem (1) f is independent of $\omega \in c_1(M)$.
(2) $f \neq 0$ implies nonexistence of KE metric.

The de nition of f was reformulated by Donaldson only using algebraic geometry in a way that can be applied to schemes. But I will not go into the detail here.

3. Known existence results

So far, I talked about obstructions. Next, I turn to **Existence Results** of K-E metrics, due to Siu, Tian, Nadel and their variants.

Siu (1988) : Enough symmetries $\Longrightarrow \exists$ K-E metric .

Tian (1987) : $\alpha(M) > \frac{m}{m+1} \Longrightarrow \exists$ K-E metric.

Nadel (1988) :
$\nexists$ of K-E metric $\Longrightarrow \exists$ of proper multiplier ideal sheaf.
i.e. $\nexists$ of proper multiplier ideal sheaf $\Longrightarrow \exists$ of K-E metric.

Demailly-Kollàr(2001) :
Simpli cation of Nadel's arguments, applications to orbifolds.

Boyer-Galicki, Kollàr :
Applications to Sasaki-Einstein metrics.

Demailly-Kollàr version of multiplier ideal sheaves
Let ψ be an ω_g-plurisubharmonic function , i.e., a real-valued upper semi-continuous function satisfying $\omega_g + \frac{\sqrt{\ 1}}{2\pi}\partial\partial\psi \geq 0$ in the current sense. The **multiplier ideal sheaf with respect to** ψ is the ideal sheaf de ned by the following presheaf

$$(U, \mathcal{I}(\psi)) = \{f \in \mathcal{O}(U) \mid \int_U |f|^2 e^{\ \ \psi} dV < \infty\}$$

where U is an open subset of M.

To prove the existence of KE metric, we consider the family of Monge-Ampére equations

$$\frac{\det(g_{ij} + \varphi_{ij})}{\det(g_{ij})} = e^{\ \ t\varphi+h}$$

for $t \in [0, 1]$.

If there is no KE metric, there exists $t_0 < 1$ such that $\{\varphi_t\}$ is the solution $\{\varphi_t\}_{0\leq t<t_0}$ and that

$$\inf_M(\varphi_t - \sup_M \varphi_t) \to -\infty$$

as $t \to t_0$. Note that solutions exist on open set of t's by Banach space implicit function theorem.

This is because of

Theorem(Yau)
If $\{\varphi_t\}$ is bounded in C^0 then $\{\varphi_t\}$ is bounded in C^3.

Thus if $\{\varphi_t\}$ is bounded in C^0 then $\{\varphi_t\}$ is uniformly bounded and equi-continuous up to second order derivatives.

Thus by Ascoli-Arzela, a suitable subsequence $\{\varphi_t\}$ converges to the solution φ_{t_0} of the Monge-Ampère equation for $t = t_0$. Then the set of t's such that a solution φ_t exits is a non-empty open and closed subset of t's. Thus we have a solution for $t = 1$. This is a contradiction because we assume there is no KE metric.

Therefore we must have

$$\inf_M(\varphi_t - \sup_M \varphi_t) \to -\infty$$

as $t \to t_0$.

Let M be a Fano manifold of dimension m.
Let G be a compact subgroup of Aut(M).
Assume that M does not a G-invariant Kähler-Einstein metric.
Let $\in (m/(m+1), 1)$. This number $(m/(m+1)$ arises from an analytic inequality for Minge-Ampère equations, called the Harnack inequality. This is too much to talk about here, and the audience should take it granted as something necessary from PDE theory.
Then there exists a sequence $\{\varphi_{t_k}\}_{k=1}^{\infty}$ such that
$t_k \to t_0$ as $k \to \infty$,
there exists $\varphi_\infty = \lim_{k\to\infty}(\varphi_{t_k} - \sup_M \varphi_{t_k})$ in L^1-topology, which is an ω_g-psh function, and
$\mathcal{I}(\ \varphi_\infty)$ is a proper multiplier ideal sheaf, i.e, $\mathcal{I}(\ \varphi_\infty)$ is neither 0 nor $\mathcal{O}_M$.

4. The relation between the MIS and the invariant f

Now I turn to the question I want to raise in **This talk** :
What is the relation between the MIS and the invariant f ?

There has been an answer to this question by Nadel stated as

Theorem (Nadel, 1995)
Suppose M does not admit a K-E metric, and let V be the support of the MIS. For $v \in \mathfrak{h}(M)$ with $f(v) = 0$ we have

$$V \not\subset \mathrm{Zero}^+(v) := \{p \in \mathrm{Zero}(v) \mid \Re((\mathrm{div}(v))(p) > 0\}.$$

Here $\mathrm{div}(v)\mathrm{vol}_g = \mathcal{L}_v \mathrm{vol}_g$. Notice that $\mathrm{div}(v)$ is independent of the choice of g along $\mathrm{Zero}(v)$.

We extend this in several ways.

to get some more informations on Fano manifolds,

to show the existence of MIS for Kähler-Ricci solitons,

to study the MIS arising from the non-convergence of Kähler-Ricci flow and study the relation between MIS and f.

So, we study three types of MIS.

KE-MIS : due to Nadel, arising from the failure of solving Monge-Ampère equations for **Kahler-Einstein** metrics by continuity method.

KRS-MIS : Arising from the failure of solving Monge-Ampère equations for **Kahler-Ricci solitons** by continuity method.

KRF-MIS : Arising from the failure of convergence of **Kahler-Ricci flow**.

Let M be a Fano manifold,
G be a compact subgroup of $\mathrm{Aut}(M)$,
T^r maximal torus of G.
For any G-invariant Kähler metric g with

$$\omega_g := \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \in c_1(M)$$

consider the Hamiltonian T^r-action with the moment map $\mu_g : M \to \mathfrak{t}^r$.
For $\xi \in \mathfrak{t}^r$ we put

$$D^{\leq 0}(\xi) := \{y \in \mu(M) \mid \ < y, \xi > \leq \ 0\}.$$

Theorem (Futaki-Sano)
Suppose M does not admit a K-E metric, and let V be the support of the KE-MIS. Let $\xi \in \mathfrak{t}^r \subset \mathfrak{h}(M)$ satisfy $f(v_\xi) > 0$ where v_ξ is the holomorphic vector eld corresponding to ξ. Then

$$\mu_g(V) \not\subset D^{\leq 0}(\xi)$$

for any G-invariant Kähler metric g in $c_1(M)$.

Corollary
Let M be the one-point blow-up of $\mathbb{CP}^2$. Then V is the exceptional divisor. This V destabilizes slope stability in the sense of Ross-Thomas.

Definition (Slope stability w.r.t. $V \subset M$):
Put $\mathcal{M}$ = blow up of $M \times \mathbb{C}$ along $V \times 0$.
The central ber is regarded as a degenaration of M.
Compute Donaldson's algebraic reformulation of f.
If it has the right sign for any V, then M is said to be **slope stable**.

<u>Outline of Proof of Theorem 2</u>

Let $h \in C^\infty(M)$ satisfy $\mathrm{Ric}_g - \omega_g = i\partial\overline{\partial}h$.
Suppose

$$\frac{\det(g_{ij} + \varphi_{ij})}{\det(g_{ij})} = e^{\ \ t\varphi+h}$$

has solutions only for $t \in [0, t_0),\ t_0 < 1$.

Then we have a MIS with support V.

Fact 1 : (Nadel, based on earlier estimates by Siu and Tian)
Let $K \subset M - V$ be a compact subset. Then

$$\int_K \omega_{g_t}^m \to 0$$

as $t \to t_0$.

Fact 2 :

$$\mu_g(p) \in D^{\leq 0}(\xi) \Longleftrightarrow (\mathrm{div}(v_\xi))(p) \geq 0$$

where

$$\mathrm{div}(v_\xi)(e^h\omega^m) = \mathcal{L}_{v_\xi}(e^h\omega^m).$$

Fact 3 :

$$\frac{t}{t-1} f(v_\xi) = \int_M \mathrm{div}(v_\xi)\omega_t^m.$$

By Fact 3 and our assumption $f(v_\xi) > 0$, we have for $t \in (\delta, t_0)$ with $t_0 < 1$

$$\int_M \mathrm{div}(v_\xi)\omega_t^m = \frac{t}{t-1} f(v_\xi) < -C$$

with $C > 0$ independent of t. Suppose $\mu_g(V) \subset D^{\leq 0}(\xi) = \{\mathrm{div}(v_\xi) \geq 0\}$.
Choose > 0 small and put

$$W\ \ := \{p \in M | \mathrm{div}(v_\xi)(p) \leq -\ \ \}.$$

Then $W\ \ \subset M - V$ compact. Apply Fact 1 to W to get

$$\int_{W_\epsilon} \omega_{g_t}^m \to 0$$

as $t \to t_0$.

But then

$$-C \geq \int_M \operatorname{div}(v_\xi)\omega_t^m = \int_{M \quad W_\epsilon} \operatorname{div}(v_\xi)\omega_t^m + \int_{W_\epsilon} \operatorname{div}(v_\xi)\omega_t^m$$
$$\geq -2 \operatorname{vol}(M,g)$$

as $t \to t_0$, a contradiction ! This completes the proof.

KRS-MIS

Let M be a Fano manifold. Choose $\omega_g \in c_1(M)$. Let $v \in \mathfrak{h}_r(M)$ be in the reductive part $\mathfrak{h}_r(M)$ of $\mathfrak{h}(M)$.

Definition : We say (g, v) is a Kähler-Ricci soliton
$\Longleftrightarrow \operatorname{Ric}(\omega_g) - \omega_g = \mathcal{L}_v(\omega_g)$.

Then $\Im(v)$ is necessarily Killing.

Start with an initial metric g^0 with $\omega_0 := \omega_{g^0} \in c_1(M)$. Let h_0 and $\ _{v,0}$ be the smooth functions such that

$$\operatorname{Ric}(\omega_0) - \omega_0 = i\partial\bar{\partial}h_0, \quad \int_M e^{h_0}\omega_0^m = \int_M \omega_0^m,$$

$$i_v\omega_0 = i\bar{\partial}\ _{v,0}, \quad \int_M e^{\theta_{v,0}}\omega_0^m = \int_M \omega_0^m.$$

Consider for $t \in [0,1]$

$$\det(g^0_{ij} + \varphi_{tij}) = \det(g^0_{ij})e^{h_0 \quad \theta_{v,0} \quad v\varphi_t \quad t\varphi_t}.$$

The solution for $t = 1$ gives the Kähler-Ricci soliton.
Zhu has shown that $t = 0$ always has a solution.
Implicit function theorem shows for some $\ > 0$, all $t \in [0, \ )$ have a solution.
Suppose we only have solutions on $[0, t_\infty)$, $t_\infty < 1$.
Let $\ _{v,g}$ satisfy

$$i_v\omega_g = i\bar{\partial}\ _{v,g}, \quad \int_M e^{\theta_{v,g}}\omega_g^m = \int_M \omega_g^m.$$

Definition :

$$f_v(w) = \int_M w(h_g - \ _{v,g})e^{\theta_{v,g}}\omega_g^m$$

This f_v is independent of g with $\omega_g \in c_1(M)$.

Theorem (Tian-Zhu) There exists a unique $v \in \mathfrak{h}_r(M)$ such that

$$f_v(w) = 0 \text{ for all } w \in \mathfrak{h}_r(M).$$

Theorem (F-Sano)

Let K be the compact subgroup such that $\mathfrak{k} \otimes \mathbb{C} = \mathfrak{h}_r(M)$. Suppose there is no KRS. Then we get MIS and its support V_v satis es

$$V_v \not\subset Z^+(w) \text{ for } \forall w \in \mathfrak{h}_r(M).$$

We can apply this to prove the existence of KRS on the one point blow-up of $\mathbb{CP}^2$.

KRF-MIS

Theorem (Phong-Sesum-Sturm)
One gets MIS from the failure of convergence of normalized Kähler-Ricci flow:

$$\frac{\partial g}{\partial t} = -\mathrm{Ric}(g) + g.$$

If we put $g_{tij} = g_{ij} + \varphi_{tij}$ the Ricci flow is equivalent to

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{\det(g_{ij} + \varphi_{tij})}{\det(g_{ij})} + \varphi_t - h_0$$
$$\varphi_0 = c_0$$

Yanir Rubinstein modi ed Phong-Sesum-Sturm's MIS using the idea of Demailly-Kollàr:

$$\varphi_t - \int_M \varphi_t \omega^m \longrightarrow \varphi_\infty \quad \text{almost psh}$$

as $t \to \infty$.

Let V_γ be the MIS for $\psi = \ \varphi_\infty$, $\ \in (\frac{m}{m+1}, 1)$, de ned by

$$(U, \mathcal{I}(\psi)) = \{\ f \ \in \mathcal{O}_M(U) \mid \int_U |f|^2 e^{\ \psi} \omega_g^m < \infty\}.$$

This MIS satis es

$$H^q(M, \mathcal{I}(\psi)) = 0 \quad \text{for} \quad \forall q > 0.$$

Yuji Sano's work:

Let M be a toric Fano manifold, and put

$$T_{\mathbb{R}} = T^m, \qquad T_{\mathbb{C}} = (\mathbb{C}\ )^m,$$
$$N_{\mathbb{R}} = J\mathfrak{t}_{\mathbb{R}}.$$

Let $W(M) = N(T_{\mathbb{C}})/T_{\mathbb{C}}$ be the Weyl group.

Theorem (Wang-Zhu) There exists a KRS (g_{KRS}, v_{KRS}).

Theorem (Sano) Suppose $\dim N_{\mathbb{R}}^{W(M)} = 1$. Let $\ _t = \exp(tv_{KRS})$, $0 < \ < 1$ and ω a $T_{\mathbb{R}}$-invariant Kähler form. Then the support of Rubinstein's KRF-MIS of exponent is equal to the support of the MIS of exponent obtained from the Kähler potentials of $\{(\ _t{}^{1})\ \omega\}$.

Using this Sano computed the support of KRF-MIS for various on some examples.