Theory of Holomorphic Curves and Related Topics

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1 Introduction

The purpose of this talk is to survey the developemnt of the Nevanlinna theory in higher dimensions, and to discuss recent results obtained by newly established Second Main Theorems (S.M.T. for abbrevation). We will also discuss a link to arithmetic problem. The plan of this talk is as follows:

(i) A. Bloch (1926), H. Cartan (1928): the target $\mathbf{P}^1(\mathbf{C}) \Rightarrow \mathbf{P}^n(\mathbf{C})$.

(ii) H.&J. Weyl (1938), A.L. Ahlfors (1941): the target $\mathbf{P}^1(\mathbf{C}) \Rightarrow \mathbf{P}^n(\mathbf{C})$.

(iii) W. Stoll (1953): the domain $\mathbf{C} \Rightarrow \mathbf{C}^m$, $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$.

(iv) A. Bloch (1926): $f : \mathbf{C} \to V$, algebraic variety.

(v) Griffiths et al. (1972): $f : \mathbf{C}^m \to V^n$ dominant ($f(\mathbf{C}^m) \supset$ open subset $\neq \emptyset$) into algebraic variety V.

(vi) Bloch-Ochiai's Theorem (1926/77): $f : \mathbf{C} \to V$ with projective algebraic V and $q(V) > \dim V$ must algebraically degenerate.

(vii) Logarithmic Bloch-Ochiai's Theorem (Noguchi 1977 1981): Unifying Borel's Theorem (1897) and Bloch-Ochiai's $\Leftarrow$ Inequality of S.M.T. type for $f : \mathbf{C} \to V$ and a divisor D on V; the extension to Kahler V by Noguchi-Winkelmann (2002).

2 Conjectures for holomrophic curves

Referring to the above mentioned results, one may pose the following:

Fundamental Conjecture for holomorhic curves. *Let X be a smooth compact algebraic variety, and let $D = \sum_j D_j$ be a reduced s.n.c. (simple normal crossing) divisor on X with smooth D_j. Then, for an algebraically non-degenerate $f : \mathbf{C} \to X$ we have*

(2.1) $$T_f(r; L(D)) + T_f(r; K_X) \quad \sum_j N_1(r; f^* D_j) + \epsilon T_f(r)||, \ {}^\forall \epsilon > 0.$$

This implies

Green-Griffiths' Conjecture (1980). *If X is a variety of (log) general type, then every entire holomeophic curve $f : \mathbf{C} \to X$ is algebraically degenerate.*

This follows from a contradictory inequality: $T_f(r) \leq \epsilon T_f(r)||$ by (2.1). Then Green-Griffiths' Conjecture implies

Kobayashi Conjecture (1970). *If $X \subset \mathbf{P}^n(\mathbf{C})$ is a general hypersurface of* $\deg X \geq 2n - 1$, *then X is Kobayashi hyperbolic.*

The above implication is supported by

Theorem 2.2 (C. Voisin (1996/98)). *Let $X \subset \mathbf{P}^n(\mathbf{C})$ be a general hypersurface of* deg $\geq 2n - 1$ *for $n \geq 3$. Then every subvariety of X is of general type.*

3 Yamanoi's *abc* Theorem (S.M.T.)

R. Nevanlinna dealt with the distibution of the roots of $f(z) - a = 0$ for a meromorhic function f and constant values a. He conjectured the same to hold for *moving targets* $a(z)$ of small order functions; this was called Nevanlinna's Conjecture.

The term "moving target" is due to W. Stoll, but such a study can go back to E. Borel, Acta 1897. Nevanlinna's Conjecture was proved with *non-truncated* counting functions (Osgood (1985), Steinmetz (1986)), and as well for moving hyperplanes of $\mathbf{P}^n(\mathbf{C})$ (M. Ru-W. Stoll (1991)).

In Acta 2004 [12], K. Yamanoi proved the best S.M.T. for meromorphic functions with respect to moving targets, where the counting functions are *truncated to level 1.* It is considered to be "*abc Theorem*" for meromorphic functions, presented by Carlo Gasbarri at *Seminaire Bourbaki* in March 2008 ("*The strong abc conjecture over function fields, after McQuillan and Yamanoï*").

Let $p : X \to S$ be a surjective morphism between smooth projective algebraic varieties with relative canonical bundle $K_{X/S}$.

Theorem 3.1 (Yamanoi, 2004/06). *Assume that*

- $\dim X/S = 1$ *;*
- *$D \subset X$ is a reduced divisor ;*
- *$f : \mathbf{C} \to X$ is algebraically nondegenerate ;*
- *$g = p \circ f : \mathbf{C} \to S$.*

Then for $^\forall \epsilon > 0$, $^\exists C(\epsilon) > 0$ such that

$$T_f(r; L(D)) + T_f(r; K_{X/S}) \leq N_1(r; f^*D) + \epsilon T_f(r) + C(\epsilon) T_g(r)||_\epsilon. \tag{3.2}$$

His method consists of the following items:

- Ahlfors' covering theory;
- Mumford's theory of the compactification of curve moduli;
- The tree theory for point configurations.

4 S.M.T. (abc) for semi-abelian varieties

Let A be an algebraic group admitting the representation,

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \text{ (abelian variety)} \to 0.$$

Such A is called a *semi-abelian variety.* The universal covering $\tilde{A} = \mathbf{C}^n, n = \dim A$.

N.B. In E. Borel's case we have that $\mathbf{P}^n(\mathbf{C}) \setminus \{n+1 \text{ hyperplanes in general possition}\} = (\mathbf{C}^*)^n$.

Let $f : \mathbf{C} \to A$ be a holomorphic curve, and set

- $J_k(A)$: the k-jet bundle over A; $J_k(A) = A \quad \mathbf{C}^{nk}$;
- $J_k(f) : \mathbf{C} \to J_k(A)$: the k-jet lift of f;
- $X_k(f)$: the Zariski closure of the image $J_k(f)(\mathbf{C})$.
- $I_k : J_k(A) = A \quad \mathbf{C}^{nk} \to \mathbf{C}^{nk}$, the projection.

Lemma 4.1 (Extended Lemma on Logarithmic Derivative, Noguchi 1977).

(i) *For a holomorphic curve* $f : \mathbf{C} \to A$*,*

$$T_{I_k \circ J_k(f)}(r) = O(\log^+(rT_f(r)))\, ||.$$

(ii) *For a holomorphic curve* $f : \mathbf{C} \to \bar{A}$ *into a compacti cation* $\bar{A}$ *of* A*,*

$$m(r; I_k \circ J_k(f)) \stackrel{\text{def}}{=} \int_{|z|=r} \log^+ \|I_k \circ J_k(f)(z)\| \frac{d\theta}{2\pi} = O(\log^+(rT_f(r)))\, ||.$$

By making use of the jet projection method developed in Noguchi 1981 and Noguchi-Ochiai 1990 we have the following S¿M.T.

Theorem 4.2 (N.-Winkelmann-Yamanoi, Acta 2002 & Forum Math. 2008, Yamanoi Forum Math. 2004). *Let* $f : \mathbf{C} \to A$ *be algebraically non-degenerate.*

(i) *Let* Z *be an algebraic reduced subvariety of* $X_k(f)$ $(k \geqq 0)$*. Then there exists a compacti cation* $\bar{X}_k(f)$ *of* $X_k(f)$ *such that for* $J_k(f) : \mathbf{C} \to \bar{X}_k(f)$

(4.3) $$T_{J_k(f)}(r; \omega_{\bar{Z}}) = N_1(r; J_k(f)^* Z) + o(T_f(r))||.$$

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$*, then*

(4.4) $$T_{J_k(f)}(r; \omega_{\bar{Z}}) = o(T_f(r))||.$$

(iii) *If* $k = 0$ *and* Z *is an e ective divisor* D *on* A*, then* $\bar{A}$ *is smooth, equivariant, and independent of* f*; furthermore,* (4.3) *takes the form*

(4.5) $$T_f(r; L(\bar{D})) = N_1(r; f^* D) + o(T_f(r; L(\bar{D})))||.$$

N.B. (1) In Noguchi-Winkelmann-Yamanoi Acta 2002, we proved (4.5) with a higher level truncated counting function $N_k(r; f^*D)$ for some special compactification of A and with a better remainder term "$O(\log^+(rT_f(r)))$".

(2) For the truncation of level 1, the remainder term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log^+(rT_f(r)))$".

The trunction level 1 in (4.3) and (4.5) implies a number of interesting applications as given in the next three sections.

5 Application I: Degeneracy theorems

Here we apply Theorem 4.2 to study the algebraic degeneracy problem for holomorphic curves into some algebraic varieties.

Theorem 5.1 (Noguchi-Winkelmann-Yamanoi, J. Math. Pure. Appl. 2007). *Let X be an algebraic variety such that*

(i) *The logarithmic irregularity* $\bar{q}(X) \geqq \dim X$*;*

(ii) *The logarithmic Kodaira dimension* $\bar{\kappa}(X) > 0$*;*

(iii) *the quasi-Albanese map $X \to A$ is proper.*

Then every holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate. Moreover, the normalization of $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a semi-abelian variety which is nite etale over a translate of a proper semi-abelian subvariety of A.

N.B. The case "$\bar{q}(X) > \dim X$" was "Log Bloch-Ochiai's Theorem" (Noguchi 1977-'81). The proof for the case "$\bar{q}(X) = \dim X$" requires the new S. M. Theorem 4.2.

Specializing X $\mathbf{P}^n(\mathbf{C})$, we have

Theorem 5.2 *Let $D = \sum_{j=1}^{q} D_j$ $\mathbf{P}^n(\mathbf{C})$ be an s.n.c. divisor. Assume that $q > n$ and $\deg D > n+1$. Then $^\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ is algebraically degenerate.*

The special case of $n = 2$, $q = 3$ was conjecture by M. Green:

Theorem 5.3 (Conjectured by M. Green, 1974). *Assume that $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C})$ omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then f is algebraically degenerate.*

N.B. Theorem 5.3 is optimal in two senses, the number of irreducible components and the total degree.

(i) The number of irreducible components is 3.

(ii) The total degree is 4.

(iii) The case of 4 lines is due to E. Borel (1897).

(iv) There exists a dominant $f : \mathbf{C}^2 \to \mathbf{P}^2(\mathbf{C}) \setminus D$ for $\deg D$ 3 (Buzzard-Lu (2000)).

It is likely that the above theorem will remain valid for singular divisors:

Question 1. Let $D = \sum_{i=1}^{q} D_i$ $\mathbf{P}^n(\mathbf{C})$ be a divisor in general position (the codimensions of intersections of D_i's decrease exactly as the number of D_i's), possibly with singularities.
Assume that $q > n$ and $\deg D > n+1$. Then, is $\bar{\kappa}(\mathbf{P}^n(\mathbf{C}) \setminus D) > 0$?

It is a very interesting question to decrease the number of irreducible components. The next step is

Question 2. Let $D = D_1 + D_2$ $\mathbf{P}^2(\mathbf{C})$ be an s.n.c. divisor with two conics D_j. Then, is an arbitrary holomorphic curve $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C}) \setminus D$ algebraically degenerate?

6 Application II: Intersection with zeros of theta

In S. Lang's "Introduction to Transcendental Numbers", Addison-Wesley, 1966, he wrote at the last paragraph of Chap. 3:

"Independently of transcendental problem one can raise an interesting question of algebraic-analytic nature, namely given a 1-parameter subgroup of an abelian variety (say Zariski dense), is its intersection with a hyperplane section necessarily non-empty, and in nite unless this subgroup is algebraic?"

In 6 years later, J. Ax (Amer. J. Math. (1972)) took this problem:

Theorem 6.1 *Let θ be a reduced theta function on $\mathbf{C}^m$ associated with a lattice Γ. Let L be a 1-dimensional affine subspace of $\mathbf{C}^m$. Then either $(\theta|L)$ is constant or has an in nite number of zeros;*

(6.2) $$|\{(\theta|L) = 0\} \cap \Delta(r)| \quad r^2.$$

N.B. There seems no reference that expicitly states the last part of the aforementioned Lang's question, $|\{(\theta|L) = 0\}/\Gamma| = \infty$ in the case where the 1-parameter subgroup is Zariski dense. In fact, we deduce this from the estimate (6.2) (Corvaja-Noguchi [4], 2009). Furthermore we can deduce more explicit, more general and more geometric statements by making of S.M. Theorem 4.2.

Theorem 6.3 *Let A be an abelian variety, let $f : \mathbf{C} \to A$ be a 1-parameter analytic subgroup with $v = f'(0)$, and let D be a reduced divisor on A with the Riemann form $H(\, , \,)$. Then*

$$N(r; f^*D) = H(v,v)\pi r^2 + O(\log r) = (1 + o(1))N_1(r; f^*D).$$

Theorem 6.4 *Let A be a semi-abelian variety of $\dim A \quad 2$, let $f : \mathbf{C} \to A$ be an algebraically non-degenerate holomorphic curve, and let D be a reduced divisor on A.*

(i) *If the stabilizer $\mathrm{St}(D) = \{a \in A; a + D = D\}$ of D is nite, then there exists an irreducible component $D' \quad D$ such that $f(\mathbf{C}) \cap D'$ is Zariski dense in D'; in particular, $|f(\mathbf{C}) \cap D| = \infty$.*

(ii) *If f is a 1-parameter subgroup and A is abelian, then $|f(\mathbf{C}) \cap D| = \infty$.*

Examples. (1) "$|\mathrm{St}(D)| < \infty$" is necessary. Set $A = (\mathbf{C}/\mathbf{Z}[i])^2$, and $f : z \in \mathbf{C} \to ([z], [z^2]) \in A$ and $D = \{0\} \quad \mathbf{C}/\mathbf{Z}[i]$. Then $|f(\mathbf{C}) \cap D| = \{0\}$.

(2) Set $A = \mathbf{C}/\mathbf{Z} \quad \mathbf{C}/\mathbf{Z}[i])$, and $f : z \in \mathbf{C} \to ([z], [z]) \in A$ and $D = \{0\} \quad \mathbf{C}/\mathbf{Z}[i]$. Then $|f(\mathbf{C}) \cap D| = \{0\}$.

7 Application III: A new unicity theorem

The following is a famous application of Nevnlinna-Cartan's S.M.T.

Theorem 7.1 (i) (Unicity Theorem, Nevanlinna $n = 1$ (1926)) *Let $f, g : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ be non-constant meromorphic functions. Asuume that there are 5 distinct points $\{a_j\}_{j=1}^5 \quad \mathbf{P}^1(\mathbf{C})$ such that* $\mathrm{Supp}\, f^*a_j = \mathrm{Supp}\, g^*a_j, 1 \quad j \quad 5$. *Then $f \equiv g$.*

(ii) (Fujimoto $n \quad 2$ (1975)) *Let $f, g : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be holomorphic curves such that at least one of them is linearly non-degenerate, and let $\{H_j\}_{j=1}^{3n+2}$ be hyperplanes in general position. If $f^*H_j = g^*H_j$ for all $1 \quad j \quad 3n + 2$, then $f \equiv g$.*

There is an arithmetic problem of the similar nature:

Erdös' Problem (1988) (Unicity problem for arithmetic recurrences). *Let x, y be positive integers. Is it true that*

$$\{p; prime, p|(x^n \quad 1)\} = \{p; prime, p|(y^n \quad 1)\}, \forall n \in \mathbf{N}$$
$$\Longleftrightarrow x = y \quad ?$$

The answer is Yes:

Theorem 7.2 (Corrales-Rodorigañez and R. Schoof, JNT 1997; inplicitly, Shinzel 1960).

(i) *Suppose that except for nitely many prime $p \in \mathbf{Z}$*

$$y^n \equiv 1 \pmod{p} \text{ whenever } x^n \equiv 1 \pmod{p}, \forall n \in \mathbf{N}.$$

Then, $y = x^h$ with some $h \in \mathbf{N}$.

(ii) *Let E be an elliptic curve de ned over a number eld k, and let $P, Q \in E(k)$. Suppose that except for nitely many prime $p \in O(k)$*

$$nQ = 0 \text{ whenever } nP = 0 \text{ in } E(k_p).$$

Then either $Q = \sigma(P)$ with $^\exists\sigma \in \mathrm{End}(E)$, or both P, Q are torsion points.

In complex analysis, K. Yamanoi proved the following striking unicity theorem in Forum Math. 2004:

Theorem 7.3 (Yamanoi's Unicity Theorem). *Let $A_i, i = 1, 2$, be abelian varieties, let $D_i \quad A_i$ be irreducible divisors such that $\mathrm{St}(D_i) = \{0\}$, and let $f_i : \mathbf{C} \to A_i$ be algebraically nondegenerate. Assume that $f_1^{-1}D_1 = f_2^{-1}D_2$ as sets. Then there exists an isomorphism $\phi : A_1 \to A_2$ such that*

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^* D_2.$$

N.B. The new point is that we can determine not only f, but the moduli point of a polarized abelian vareity (A, D) through the distribution of $f^{-1}D$ by an algebraically nondegenerate $f : \mathbf{C} \to A$.

We want to generalize this to semi-abelian varieties to have a uniformized theory.

Let $A_i, i = 1, 2$ be semi-abelian varieties:

$$0 \to (\mathbf{C}^*)^{t_i} \to A_i \to A_{0i} \to 0.$$

Let D_i be an irreducible divisor on A_i such that $\mathrm{St}(D_i) = \{0\}$ just for simplicity; this is not restrictive.

Theorem 7.4 (Corvaja-Noguchi [4], preprint 2009). *Let $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ be non-degenerate holomorphic curves. Assume that*

$$\underline{\mathrm{Supp}\, f_1^* D_1}_\infty \quad \underline{\mathrm{Supp}\, f_2^* D_2}_\infty \quad (\textit{as germs at } \infty), \tag{7.5}$$

and that there is a positive constant c such that

$$cN_1(r, f_1^* D_1) \quad N_1(r, f_2^* D_2)||. \tag{7.6}$$

Then there exists a nite etale morphism $\phi : A_1 \to A_2$ such that

$$\phi \circ f_1 = f_2, \quad D_1 \quad \phi^* D_2.$$

If equality holds in (7.5), then ϕ is an isomorphism and $D_1 = \phi^* D_2$.

N.B. Assumption (7.6) is necessary by example.

The following is immediate from Theorem 7.4.

Corollary 7.7 (i) *Let* $f : \mathbf{C} \to \mathbf{C}^*$ *and* $g : \mathbf{C} \to E$ *with an elliptic curve* E *be holomorphic and non-constant. Then*

$$\underline{f^{-1}\{1\}}_\infty \neq \underline{g^{-1}\{0\}}_\infty .$$

(ii) *If* $\dim A_1 \neq \dim A_2$ *in Theorem 7.4, then*

$$\underline{f_1^{-1} D_1}_\infty \neq \underline{f_2^{-1} D_2}_\infty .$$

N.B. The di erence of value distribution property caused by the infinite cyclic quotient $\langle \tau \rangle$ cannot be recovered by the choices of f and g even thoough they are allowed to be arbitrarily transcendental:

$$\begin{array}{ccc} \mathbf{C} & \stackrel{f}{\to} & \mathbf{C}^* \\ & \searrow_{g} & \downarrow /\langle \tau \rangle \\ & & E \end{array}$$

8 Arithmetic Recurrence

It is natural to expect an analogue in arithmetic as in the new unicity Theorem 7.4.

For the *linear tori* we can prove such a result, but the case of a general semi-abeian variety is left to be an open conjecture. We use the following notaion:

- $\mathcal{O}_S$ be a ring of S-integers in a number field k;
- $\mathbf{G}_1$, $\mathbf{G}_2$ be linear tori;
- D_i be reduced divisors on $\mathbf{G}_i$ defined over k;
- $\mathcal{I}(D_i)$ be the defining ideals;
- every irreducible component of D_i has a finite stabilizer, and $\mathrm{St}(D_2) = \{0\}$.

Theorem 8.1 (Corvaja-N., preprint 2009). *Let* $g_i \in \mathbf{G}_i(\mathcal{O}_S)$ *be elements generating Zariski-dense subgroups. Suppose that for in nitely many* $n \in \mathbf{N}$,

$$(g_1^n)^* \mathcal{I}(D_1) \supset (g_2^n)^* \mathcal{I}(D_2). \tag{8.2}$$

Then there exists a nite etale morphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$*, de ned over* k*, and* $^\exists h \in \mathbf{N}$ *such that* $\phi(g_1^h) = g_2^h$ *and* $D_1 \quad \phi^*(D_2)$.

N.B. (i) Theorem 8.1 is deduced from Corvaja-Zannier, Invent. Math. 2002.

(ii) By an example we cannot take $h = 1$ in general.

(iii) By an example, the condition on the stabilzers of D_1 and D_2 cannot be omitted.

(iv) Note that inequality (inclusion) (8.2) of ideals is assumed only for an infinite sequence of n, not necessarily for all large n. On the contrary, we need the inequality of ideals, not only of their *supports*, i.e. of the primes containing the corresponding ideals.

(v) One might ask for a similar conclusion assuming only the inequality of supports. There is some answer for it, but it is in a weaker form.

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