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THE LIMIT OF THE FOURIER-MUKAI TRANSFORM FOR TORUS RANK ONE DEGENERATIONS

GERARD VAN DER GEER

ABSTRACT. We consider what happens with the Fourier-Mukai transform if an abelian variety degenerates.

The topic of this lecture is joint work with Alexis Kouvidakis from the University of Crete at Iraklion. The cohomology and the Chow ring of a principally polarized abelian variety possess a rich and intricate structure that arises from the group law on the abelian variety. We consider the Chow ring $\mathrm{CH}^*_{\mathbb{Q}}(X)$ of a principally polarized abelian variety X of dimension g over an algebraically closed field k. Coefficients are taken from the rational numbers. This ring is graded by codimension $\mathrm{CH}^*_{\mathbb{Q}}(X) = \quad {}_i \mathrm{CH}^i_{\mathbb{Q}}(X)$ and it carries an intersection product denoted by $(x, y) \mapsto x \quad y$. But there is another product structure, the Pontryagin product, denoted by $(x, y) \mapsto x \star y$. This gives a second structure of a commutative ring on $\mathrm{CH}^*_{\mathbb{Q}}(X)$.

These two ring structures are related by a map called the Fourier-Mukai transform $F : \mathrm{CH}^*_{\mathbb{Q}}(X) \to \mathrm{CH}^*_{\mathbb{Q}}(X^t)$ with X^t the dual abelian variety. The map F is defined by $x \mapsto p_{2*}(e^{c_1(P)} \quad p_1^*(x))$, where p_1 and p_2 are the two projections of the product $X \quad X^t$ and P is the Poincare bundle on $X \quad X^t$. The principal polarization allows us to identify X and X^t. We can then interpret this Fourier-Mukai transform as an isomorphism from the one ring structure to the other one: F induces an isomorphism of $(\mathrm{CH}^*_{\mathbb{Q}}(X), \ )$ with $(\mathrm{CH}^*_{\mathbb{Q}}(X), \star)$ interchanging the usual product with the Pontryagin product. But there is more to it. The integers act as endomorphisms on X, hence $\mathbb{Z} \quad \mathrm{End}(X)$ acts on $\mathrm{CH}^*_{\mathbb{Q}}(X)$. Beauville used this (see [1]) to define a second grading $\mathrm{CH}^i_{\mathbb{Q}}(X) = \quad {}_j \mathrm{CH}^i_{(j)}(X)$ by

$$\mathrm{CH}^i_{(j)}(X) := \{x \in \mathrm{CH}^i_{\mathbb{Q}}(X) : n^*x = n^{2i-j}x \quad \text{for all } n \in \mathbb{Z}\}.$$

Then under the action of F the space $\mathrm{CH}^i_{(j)}(X)$ is sent bijectively to $\mathrm{CH}^{g-i+j}_{(j)}(X)$. This implies $i \quad g \quad j \quad i$. This very rich structure on $\mathrm{CH}^*_{\mathbb{Q}}(X)$ is inherited if we work modulo algebraic equivalence and consider the ring $A(X) = \mathrm{CH}^*_{\mathbb{Q}}(X)/\sim_{\mathrm{alg}}$. We can write $A(X) = \quad {}_{i,j} A^i_{(j)}$.

The Fourier-Mukai transform acts as a magic wand in bringing about all this intricate structure. Now often one can learn a lot from letting a variety degenerate. If we want to do that in our case we need to know what happens with the Fourier-Mukai transform in case our abelian variety degenerates. The easiest case of degeneration is the case of a torus rank 1 degeneration. We look at a semi-abelian variety $\mathcal{X} \to S$ over a discrete valuation ring with residue field k so that the generic fibre is a principally polarized abelian variety and the special fibre X_0 is a

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semi-abelian variety of torus rank 1. This means that we have an exact sequence

$$1 \to \mathbb{G}_m \to X_0 \to B \to 0,$$

where B is a principally polarized abelian variety. So X_0 is an extension of B by the multiplicative group $\mathbb{G}_m$ and the extension class is an element $\beta \in B^t = B$.

In order to be able to speak meaningfully about the limits of our cycles we need to compactify the special fibre. Therefore, we assume that $\mathcal{X}$ has a compactification $\bar{\mathcal{X}}$ over S whose special fibre $\bar{X}_0$ has the following structure. Its normalization is a $\mathbb{P}^1$-bundle $\mathbb{P} = \mathbb{P}(O_B \quad J)$ over B with J the line bundle $O(\Theta \quad \Theta_\beta)$ with the theta divisor Θ defining the polarization on B. Then the non-normal variety $\bar{X}_0$ is obtained by gluing the two natural sections of the $\mathbb{P}^1$-bundle by a translation over the element β that defines the extension class.

Now let c_η be an algebraic cycle on X_η and take the Fourier-Mukai transform $\varphi_\eta := F(c_\eta)$. It is natural to consider the limit cycle (specialization) φ_0 of φ_η. The first question is: *what is the limit φ_0 of φ_η?*

The Chow ring of $\mathbb{P}$ is an extension $\mathrm{CH}^*(B)[\lambda]/(\lambda^2 \quad \lambda \;\, q^*c_1(J))$ with $\lambda = c_1(O_{\mathbb{P}}(1))$ and $q : \mathbb{P} \to B$ the natural projection of the $\mathbb{P}^1$-bundle. We denote by c_0 the specialization of the cycle c_η on $\bar{X}_0$. We write c_0 as $\nu_*(\gamma)$ with $\gamma = q^*z + q^*w \;\, \lambda$, where $\nu : \mathbb{P} \to \bar{X}_0$ denotes the normalization map. We first state the result for algebraic equivalence (denoted by $\stackrel{a}{=}$) since it is simple.

Theorem 0.1. *Let c_η be a cycle on X_η with $c_0 = \nu_*(q^*z + q^*w \;\, \lambda)$. The limit φ_0 is up to algebraic equivalence given by*

$$\varphi_0 \stackrel{a}{=} \nu_*(q^*F_B(w) \quad q^*F_B(z) \;\, \lambda)$$

with F_B the Fourier-Mukai transform of B.

Modulo rational equivalence the answer is more involved.

Theorem 0.2. *The limit φ_0 of the Fourier-Mukai transform $\varphi_\eta = F(c_\eta)$ up to rational equivalence is given by $\varphi_0 = \nu_*(q^*a + q^*b \;\, \lambda)$ with*

$$a = F_B(w) + \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(\;\,1)^m}{(n+2)!} F_B[(z + w \;\, c_1(J)) \;\, c_1^m(J)] \;\, c_1^{n-m+1}(J)$$

and

$$b = \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(\;\,1)^m}{(n+2)!} F_B[(((\;\,1)^{n+1} \quad 1)z \quad w \;\, c_1(J)) \;\, c_1^m(J)] \;\, c_1^{n-m}(J)\,.$$

One can apply this to prove non-vanishing results. In general it is difficult to prove non-vanishing of algebraic cycles when not by homological arguments. These methods fail when we consider cycles that are homologically trivial, like elements in $A^i_{(j)}$ with $j > 0$. The first and classic non-vanishing result is the theorem of Ceresa ([2]) that says that the cycles C and $C^- = (\;\,1_{\mathrm{Jac}(C)})^*C$ on the Jacobian of a general curve C of genus $g \quad 3$ over $\mathbb{C}$ are not algebraically equivalent.

It is natural to try to use the above results to prove non-vanishing. For example, suppose that $c_\eta = \sum c^{(j)}$ with $c^{(j)} \in A^i_{(j)}(X)$ with corresponding decomposition $\varphi_\eta = \sum \varphi^{(j)}$ with $\varphi^{(j)} \in A^{g-i+j}(X)$. Then we have:

Corollary 0.3. *Suppose $\varphi_0^{(j)} \neq 0$ with $\varphi_0^{(j)}$ the codimension $g \quad i + j$-part of φ_0. Then $c^{(j)} \neq 0$ (modulo algebraic equivalence).*

As an example we take for the degenerating abelian variety the 5-dimensional Picard variety of the Fano surface Σ of lines on a cubic threefold that degenerates to a generic one-nodal cubic threefold. In this case the special fibre is the $\mathbb{G}_m$-extension of the Jacobian of a general curve C of genus 4 and the extension class β is given by the di erence of the two g_3^1's on C. After choosing a base point $s_0 \in \Sigma$ we can embed the Fano surface Σ in $\mathrm{Pic}^0(\Sigma)$ by sending s to $[D_s \quad D_{s_0}]$ with D_s the divisor of lines on the cubic threefold that intersect the line corresponding to s. The cycle class of the Fano surface Σ of the general fibre in $\mathrm{Pic}^0(\Sigma)$ has a Beauville decomposition

$$[\Sigma] = \Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)}$$

with $\Sigma^{(j)} \in A^3_{(j)}(X)$. The claim is that $\Sigma^{(1)}$ is not algebraically equivalent to 0 because it degenerates to the class with Fourier-Mukai transform

$$\varphi_0^{(1)} = \nu_*(q^*[(F_B(C^{(0)}) \quad F_B(C^{(1)})] \quad q^* F_B(C^{(1)}) \quad \lambda)$$

with $C = C^{(0)} + C^{(1)}$ the Beauville decomposition of C in $A^*(\mathrm{Jac}(C))$ and this is not zero since we know by Ceresa's classical result [2] that for sufficiently general C the class $C^{(1)}$ is not zero. We hope that other and more important applications will be found.

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Korteweg-de Vries Instituut, Universiteit van Amsterdam, Postbus 94248, 1090 GE Amsterdam, The Netherlands

E-mail address: `geer@science.uva.nl`