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The supremum of Newton polygons of p -divisible groups with a given p -kernel type

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Abstract

We show that there exists the supremum of Newton polygons of p -divisible groups with a given p -kernel type, and provide an algorithm determining it. This is an unpolarized analogue of Oort conjecture related to determining the generic Newton polygon of each Ekedahl-Oort stratum in the moduli space of principally polarized abelian varieties.

1 Introduction

Let $\mathcal{A}_g$ be the moduli space (over $\mathbb{Z}$) of principally polarized abelian varieties of dimension g . It is well-known that

$$\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H},$$

where $\mathbb{H}$ is the Siegel upper half space

$$\mathbb{H} = \{Z \in \mathrm{M}_g(\mathbb{C}) \mid Z = {}^t Z, \mathrm{Im} Z > 0\}.$$

From now on we write $\mathcal{A}_g := \mathcal{A}_g \otimes \mathbb{F}_p$. Here is an expectation (so-called a paving of $\mathcal{A}_g$):

- (1) There exists a natural decomposition of $\mathcal{A}_g$ into finitely many locally closed subschemes:

$$\mathcal{A}_g = \coprod_{\nu} \mathcal{T}_{\nu}$$

- (2) Each $\mathcal{T}_{\nu}$ can be beautifully described.

Here this decomposition should be a decomposition by natural invariants of p -divisible groups of abelian varieties.

Let S be a connected scheme. Let p be a prime number. A p -divisible group over S of height h is an inductive system

$$X = \varinjlim_{i \in \mathbb{N}} X_i, \quad X_i \subset X_{i+1}$$

of finite locally free group schemes X_i of rank p^{ih} over S such that

$$X_i = X_{i+1}[p^i],$$

where $G[N] := \text{Ker}(N : G \rightarrow G)$. For example

$$\mathbb{Q}_p/\mathbb{Z}_p, \quad \mathbb{G}_m[p^\infty], \quad A[p^\infty]$$

with an abelian scheme A over S , where

$$G[p^\infty] = \varinjlim_{i \in \mathbb{N}} G[p^i].$$

Let k be an algebraically closed field of characteristic p . We have two invariants of a p -divisible group X over k .

- (1) $\mathcal{N}(X)$:= the isogeny class (= Newton polygon) of X ,
Dieudonné-Manin classification (1963);
- (2) $\mathcal{E}(X)$:= the isomorphism class of $X[p]$,
Kraft's classification (1975).

We want to estimate $\mathcal{N}(X)$ from $\mathcal{E}(X)$.

Today's aim: Let w be any p -kernel type. We give a combinatorial algorithm determining the Newton polygon $\xi(w)$ satisfying

$$\begin{aligned} \forall X, \quad \mathcal{E}(X) = w &\implies \mathcal{N}(X) \prec \xi(w), \\ \exists Y, \quad \mathcal{E}(Y) = w &\text{ and } \mathcal{N}(Y) = \xi(w). \end{aligned}$$

The existence of the optimal upper bound $\xi(w)$ is non-trivial.
The (principally) polarized case - Sp_{2g} (2007):

The problem obtained by replacing “ p -divisible group” by “principally polarized p -divisible group”. We use the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties and the theory on stratifications on $\mathcal{A}_g$.

The unpolarized case - GL_r (Today):

No natural moduli space!

Instead we treat families of p -divisible groups and families of F -zips, and consider stratifications on those.

Geometric meaning in the polarized case:

$$\begin{aligned}\mathcal{A}_g &= \coprod_{\xi} \mathcal{W}_{\xi}^0 && : \text{Newton polygon stratification,} \\ \mathcal{A}_g &= \coprod_w \mathcal{S}_w && : \text{Ekedahl-Oort stratification,} \\ \mathcal{W}_{\xi}^0 &:= \{A \in \mathcal{A}_g \mid \mathcal{N}(A) = \xi\}, \\ \mathcal{S}_w &:= \{A \in \mathcal{A}_g \mid \mathcal{E}(A) = w\}.\end{aligned}$$

Open problem:

- (1) When $\mathcal{W}_{\xi}^0 \cap \mathcal{S}_w = \emptyset$?
- (2) Can $\mathcal{W}_{\xi}^0 \cap \mathcal{S}_w$ be beautifully described?

Today's aim in the pol. case $\iff$ When $\mathcal{S}_w \subset \overline{\mathcal{W}_{\xi}^0}$?

2 Preliminaries

2.1 The Dieudonné theory

Let K be a perfect field. Let A_K denote the ring

$$W(K)[\mathcal{F}, \mathcal{V}] / (\mathcal{F}a - a^{\sigma}\mathcal{F}, \mathcal{V}a^{\sigma} - a\mathcal{V}, \mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p),$$

where σ is the Frobenius map $W(K) \rightarrow W(K)$.

Definition 2.1. A Dieudonné module (DM) over K is a left A_K -module which is finitely generated as a $W(K)$ -module.

Theorem 2.2 (Dieudonné theory). *There are categorical equivalences:*

$$\begin{aligned}\mathbb{D} : \{p\text{-divisible groups}/K\} &\simeq \{DM/K \text{ free as } W(K)\text{-mod.}\} \\ \mathbb{D} : \{\text{fin. } p\text{-group sch.}/K\} &\simeq \{DM/K \text{ of fin. length}\}\end{aligned}$$

2.2 Minimal p -divisible groups

For a pair (m, n) of coprime non-negative integers, we define a p -divisible group $H_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p e_i$$

with $\mathcal{F}e_i = e_{i+n}$, $\mathcal{V}e_i = e_{i+m}$ and $e_{i+m+n} = pe_i$ ($i \in \mathbb{Z}_{\geq 0}$).

Let ξ be a Newton polygon $\sum_{i=1}^t (m_i, n_i)$ (a formal sum).

Definition 2.3. A *minimal p -divisible group* of ξ is the p -divisible group

$$H(\xi) = \bigoplus_{i=1}^t H_{m_i, n_i}.$$

2.3 Newton polygons

A Newton polygon $\xi = \sum_{i=1}^t (m_i, n_i)$ is regarded as a lower convex polygon with $(m_i + n_i)$ slopes $\lambda_i := m_i / (m_i + n_i)$ ($\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{t-1} \leq \lambda_t$).

$$\zeta \prec \xi \iff \forall \text{point of } \zeta \text{ is above or on } \xi.$$

Let X be a p -divisible group over $k = \bar{k}$. We write $\mathcal{N}(X) = \xi$ if X is isogenous to $H(\xi)$.

Theorem 2.4 (Dieudonné-Manin classification). *We have a natural bijection:*

$$\mathcal{N} : \{p\text{-divisible groups over } k\} / \text{isog.} \simeq \{\text{Newton polygons}\}.$$

We call ξ symmetric if $\lambda_i + \lambda_{t+1-i} = 1$. Note $\mathcal{N}(A) := \mathcal{N}(A[p^\infty])$ for $A \in \mathcal{A}_g(k)$ is symmetric.

2.4 Final elements in the Weyl groups

Let W_G denote the Weyl group of $G = GL_r$ or Sp_{2g} .

$$\begin{aligned} W_{GL_r} &= \text{Aut}\{1, \dots, r\}, \\ W_{Sp_{2g}} &= \{w \in W_{GL_{2g}} \mid w(i) + w(2g+1-i) = 2g+1\}. \end{aligned}$$

We define a subset ${}^J W_G$ of W_G by

$$\begin{aligned} {}^J W_{GL_r} &:= \left\{ w \in W_{GL_r} \mid \begin{array}{l} w^{-1}(1) < \dots < w^{-1}(d), \\ w^{-1}(d+1) < \dots < w^{-1}(r) \end{array} \right\}, \\ {}^J W_{Sp_{2g}} &:= \{w \in W_{Sp_{2g}} \mid w^{-1}(1) < \dots < w^{-1}(g)\}, \end{aligned}$$

where $J = \{s_1, \dots, s_{r-1}\} \setminus \{s_d\}$ resp. $J = \{s_1, \dots, s_g\} \setminus \{s_g\}$.

An element of ${}^J W_G$ is called a final element of W_G .

A BT_1 over S is a finite locally free group scheme G over S such that

$$\begin{aligned} \text{Ker}(F : G \rightarrow G^{(p)}) &= \text{Im}(V : G^{(p)} \rightarrow G), \\ \text{Im}(F : G \rightarrow G^{(p)}) &= \text{Ker}(V : G^{(p)} \rightarrow G). \end{aligned}$$

Let k be an algebraically closed field of characteristic p .

Theorem 2.5 (Kraft, Oort, Moonen, Wedhorn).

$$\begin{aligned} \{BT_1 \text{ 's over } k \text{ of rank } p^r \text{ and dimension } d\}_{/\simeq} &\simeq {}^J W_{GL_r} \\ \{polarized \text{ } BT_1 \text{ 's over } k \text{ of rank } p^{2g}\}_{/\simeq} &\simeq {}^J W_{Sp_{2g}}. \end{aligned}$$

Note that G over k is a BT_1 if and only if $G \simeq X[p]$ for a p -divisible group X over k . A polarization on G is a non-degenerate alternating form $\mathbb{D}(G) \otimes_k \mathbb{D}(G) \rightarrow k$ satisfying $\langle \mathcal{F}x, y \rangle = \langle x, \mathcal{V}y \rangle^\sigma$ for all $x, y \in \mathbb{D}(G)$.

3 The polarized case

3.1 Stratifications on $\mathcal{A}_g$

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g in characteristic p .

$$\begin{aligned}\mathcal{A}_g &= \coprod_{\xi} \mathcal{W}_{\xi}^0 && : \text{Newton polygon stratification,} \\ \mathcal{A}_g &= \coprod_w \mathcal{S}_w && : \text{Ekedahl-Oort stratification,}\end{aligned}$$

where we define

$$\begin{aligned}\mathcal{W}_{\xi}^0 &:= \{A \in \mathcal{A}_g \mid \mathcal{N}(A) = \xi\}, \\ \mathcal{S}_w &:= \{A \in \mathcal{A}_g \mid \mathcal{E}(A) = w\}.\end{aligned}$$

3.2 Oort's conjecture

Conjecture 3.1 (Oort).

$$\mathcal{W}_{\xi}^0 \cap \mathcal{S}_w \neq \emptyset \quad \Rightarrow \quad \mathcal{Z}_{\xi} \subset \overline{\mathcal{S}_w}$$

Here $\mathcal{Z}_{\xi}$ is defined to be

$$\mathcal{Z}_{\xi} = \{A \in \mathcal{A}_g \mid A[p^{\infty}]_{\Omega} \simeq H(\xi)_{\Omega} \text{ for some } \Omega = \overline{\Omega}\},$$

which is shown to be a closed subset of $\mathcal{W}_{\xi}^0$. We call $\mathcal{Z}_{\xi}$ the central stream of ξ . Oort showed

$$\begin{aligned}\mathcal{Z}_{\xi} &= \{A \in \mathcal{A}_g \mid A[p]_{\Omega} \simeq H(\xi)[p]_{\Omega} \text{ for some } \Omega = \overline{\Omega}\} \\ &= \mathcal{S}_{\mu(\xi)},\end{aligned}$$

where $\mu(\xi)$ is the p -kernel type $\mathcal{E}(H(\xi))$ of $H(\xi)$.

3.3 Irreducibility of Ekedahl-Oort strata

The irreducibility of $\mathcal{S}_w$ depends on whether $\mathcal{S}_w \subset \mathcal{W}_{\sigma}$.

Theorem 3.2 (Ekedahl - van der Geer). *$\mathcal{S}_w$ is irreducible if $\mathcal{S}_w \not\subset \mathcal{W}_{\sigma}$.*

Theorem 3.3 (H., to appear in J. Alg. Geom.). *$\mathcal{S}_w$ is reducible for $p \gg 0$ if $\mathcal{S}_w \subset \mathcal{W}_{\sigma}$.*

Definition 3.4. The *generic Newton polygon* of $\mathcal{S}_w$ is defined to be

$$\xi(w) = \text{Newton polygon of a (every) generic point of } \mathcal{S}_w.$$

By Grothendieck-Katz, $\xi(w)$ is the optimal upper bound:

$$\begin{aligned} \forall X, \quad \mathcal{E}(X) = w &\implies \mathcal{N}(X) \prec \xi(w), \\ \exists Y, \quad \mathcal{E}(Y) = w &\quad \& \quad \mathcal{N}(Y) = \xi(w). \end{aligned}$$

3.4 Results

Theorem 3.5 (H., to appear in Ann. Inst. Fourier). *For any $w \in {}^J W_{\mathrm{Sp}_{2g}}$, we have*

$$\xi(w) = \max_{\prec} \{ \xi \mid \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w} \}.$$

This gives a combinatorial algorithm determining the generic Newton polygon $\xi(w)$ of $\mathcal{S}_w$. Recall that $\mathcal{Z}_\xi = \mathcal{S}_{\mu(\xi)}$, where $\mu(\zeta)$ is the p -kernel type of $H(\zeta)$.

Theorem 3.6 (H., Asian J. Math. (2009)).

$$\mathcal{Z}_\zeta \subset \overline{\mathcal{Z}_\xi} \iff \zeta \prec \xi.$$

Corollary 3.7. *Oort's conjecture is true: $\mathcal{W}_\zeta^0 \cap \mathcal{S}_w \neq \emptyset \implies \mathcal{Z}_\zeta \subset \overline{\mathcal{S}_w}$.*

4 The unpolarized case

4.1 Main results

Theorem 4.1 (H.). *Let $w \in {}^J W_{\mathrm{GL}_r}$. The optimal upper bound $\xi(w)$ exists, and*

$$\xi(w) = \max_{\prec} \{ \xi \mid \mu(\xi) \subset w \}.$$

This gives a combinatorial algorithm determining $\xi(w)$. See below for what $\subset$ means. Again recall $\mu(\xi) = \mathcal{E}(H(\xi))$.

Theorem 4.2 (H.). $\mu(\zeta) \subset \mu(\xi) \iff \zeta \prec \xi$.

Corollary 4.3 (The unpolarized analogue of Oort's conjecture). *If there exists a p -divisible group X with Newton polygon ζ and p -kernel type w , then we have $\mu(\zeta) \subset w$.*

Because $\zeta \prec \xi(w)$ and therefore $\mu(\zeta) \subset \mu(\xi(w)) \subset w$.

4.2 F -zips and displays

Let S be an $\mathbb{F}_p$ -scheme. Let σ be the absolute Frobenius on S . For any $\mathcal{O}_S$ -module M we write $M^{(p)} = \mathcal{O}_S \otimes_{\sigma, \mathcal{O}_S} M$.

Definition 4.4 (Moonen-Wedhorn). An F -zip over S is a quintuple $Z = (N, C, D, \varphi, \dot{\varphi})$ consisting of locally free $\mathcal{O}_S$ -module N and $\mathcal{O}_S$ -submodules C, D of N which are locally direct summands of N , and isomorphisms $\varphi : (N/C)^{(p)} \rightarrow D$ and $\dot{\varphi} : C^{(p)} \rightarrow N/D$.

If $S = \text{Spec}(K)$ with a perfect field K , then

$$\{\text{BT}_1 \text{ 's over } K\} \xrightarrow{\sim} \{F\text{-zips over } K\}$$

sending G to $(\mathbb{D}(G), \mathcal{V}N, \mathcal{F}N, \mathcal{F}, \mathcal{V}^{-1})$.

From now on we write $W = W_{\text{GL}_r}$ and ${}^J W = {}^J W_{\text{GL}_r}$.

Definition 4.5. Let $w, w' \in {}^J W$. We say $w \subset w'$ if there is an F -zip over a valuation ring such that the special fiber is of type w and the generic fiber is of type w' .

Theorem 4.6 (Wedhorn). (1) $\subset$ gives an ordering on ${}^J W$.

(2) There exists a combinatorial algorithm determining whether $w \subset w'$ for concretely given w and w' .

One can show that

Lemma 4.7. Let $w, w' \in {}^J W_{\text{GL}_r}$. If $w \subset w'$, then we have $\xi(w) \prec \xi(w')$.

Let R be a commutative ring. Let F and V be the Frobenius and Verschiebung on $W(R)$. Put $I_R = {}^V W(R)$.

A display over R is a quadruple $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ of

- (i) P : a finitely generated projective $W(R)$ -module;
- (ii) Q : a submodule of P such that $\exists$ decomposition $P = L \oplus T$ such that $Q = L \oplus I_R T$;
- (iii) $\mathcal{F} : P^{(p)} \rightarrow P$ and $\mathcal{V}^{-1} : Q^{(p)} \rightarrow P$: $W(R)$ -linear maps.

Theorem 4.8 (Zink). Assume R is an excellent local ring or of finite type over a field of char. p . Then

$$\{\text{nilpotent displays over } R\} \simeq \{\text{formal } p\text{-div. gp. over } R\}.$$

An F -zip over R is the mod I_R -reduction of a display over R .

4.3 The existence of $\xi(w)$

In the polarized case, the existence of $\xi(w)$ follows from the irreducibility of Ekedahl-Oort strata. Instead we prove

Lemma 4.9. *There exists an irreducible catalogue of p -divisible groups with a given p^m -kernel type: Let $m \in \mathbb{N}$, and let u be a p^m -kernel type. There exists a p -divisible group $\mathcal{X}$ over an irreducible scheme S of finite type over k such that*

- (1) *every geometric fiber $\mathcal{X}_s$ is of p^m -kernel type u ;*
- (2) *For any p -divisible group X with p^m -kernel type u , there exists a geometric point $s \in S$ such that $X \simeq \mathcal{X}_s$.*

This (for $m = 1$) proves that the optimal upper bound $\xi(w)$ exists. Indeed the Newton polygon of the generic fiber of $\mathcal{X}$ satisfies all the properties of $\xi(w)$.

Proof. Let $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ be a display over k , and $P = L \oplus T$ be a normal decomposition. Let

- (a) $G := \mathrm{GL}(P)$ the general linear group over $W(k)$;
- (b) H : the paraholic subgroup of G stabilizing Q ;
- (c) $\mathcal{D}_m$: the group scheme over k representing the functor

$$\mathrm{Alg}_k \rightarrow \mathrm{Set} : R \mapsto G(W_m(R));$$

- (d) $\mathcal{H}_m$: the group scheme over k representing the functor

$$\mathrm{Alg}_k \rightarrow \mathrm{Set} : R \mapsto H(W_m(R)).$$

We have that $\mathcal{D}_m$ and $\mathcal{H}_m$ are connected smooth affine group schemes over k , see Vasiu [J. Alg. Geom. (2008)]. For any truncated Barsotti-Tate group of level m (BT_m) with codim. c and dim. d , its Dieudonné module is written as $(P/p^m P, g\mathcal{F}, \mathcal{V}g^{-1})$ for some $g \in \mathcal{D}_m$. Let

$$\mathbf{BT}_m(k) = \{\mathrm{BT}_m \text{ over } k \text{ of codim. } c \text{ and dim. } d\} / \simeq.$$

Vasiu introduced an action:

$$\mathbb{T}_m : \mathcal{H}_m \times_k \mathcal{D}_m \longrightarrow \mathcal{D}_m,$$

and showed that

$$\{\mathbb{T}_m\text{-orbits}\} \simeq \mathbf{BT}_m(k).$$

Now we can construct an irreducible catalogue of p -divisible groups with p^m -kernel type u .

Choose an integer $N \geq m$ so that $X[p^N] \simeq Y[p^N]$ implies $X \simeq Y$ for any p -divisible groups X and Y over k . Let π be the natural map $\mathcal{D}_N \rightarrow \mathcal{D}_m$, and let τ be a section of $\mathcal{D} \rightarrow \mathcal{D}_N$. Let $\mathbb{O}_u$ be the $\mathbb{T}_m$ -orbit associated to u . Since $\mathcal{H}_m$ is irreducible, $\mathbb{O}_u$ is irreducible. Since π is smooth with connected fibers, $\pi^{-1}(\mathbb{O}_u)$ is also irreducible. Let S be the image of $\pi^{-1}(\mathbb{O}_u)$ by τ . Then S is irreducible and of finite type over k . By Zink's display theory, we have a p -divisible group $\mathcal{X}$ over S . Clearly $\mathcal{X}$ satisfies the required properties.

4.4 Outline of the proof (1st slope theory and induction)

Let $w \in {}^J W_{\text{GL}_r}$. Set $\nu_w(i) = \sharp\{a \leq i \mid w(a) > d\}$. We define a map

$$\Psi_w : \{1, \dots, r\} \rightarrow \{1, \dots, r\}$$

by $\Psi_w(i) = d + i$ if $w(i) = i$ and $\Psi_w(i) = \nu_w(i)$ otherwise. Let

$$\begin{aligned} \mathcal{D} &= \text{Im } \Psi_w^k \quad \text{for } k \gg 0, \\ \mathcal{C} &= \mathcal{D} \cap \{d + 1, \dots, r\}. \end{aligned}$$

Theorem 4.10 (H., J. Pure Appl. Algebra (2009)). (1) *The last slope of $\xi(w)$ is equal to $\rho(w) := \sharp\mathcal{C}/\sharp\mathcal{D}$.*

$$(2) \quad \rho(w) = \max\{m/(m+n) \mid H_{m,n}[p] \xrightarrow{\exists} G_w\}.$$

The first slope $\lambda(w)$ of $\xi(w)$ is equal to $1 - \rho(w^\vee)$.

$$\lambda(w) = \min\{m/(m+n) \mid G_w \xrightarrow{\exists} H_{m,n}[p]\}.$$

For the polarized case, see H., J. Alg. Geom. (2007).

To show the main theorem, it suffices to show

Proposition 4.11. *Assume that w is not minimal. Then there exists a non-constant family of isogenies of p -divisible groups*

$$H(\xi(w))_S \longrightarrow \mathcal{X}$$

over S such that the isomorphism type of $\mathcal{X}_s[p]$ is w for every geometric point s of S .

The main theorem follows from this proposition:

Proof of Prop. $\Rightarrow$ the main theorem. We first claim that the main theorem

$$\xi(w) = \max\{\zeta \mid \mu(\zeta) \subset w\} \tag{1}$$

is equivalent to

$$\mu(\xi(w)) \subset w. \quad (2)$$

Obviously (1) implies (2). Conversely suppose (2). Put $\star = \{\zeta \mid \mu(\zeta) \subset w\}$. Clearly (2) says $\xi(w) \in \star$. Let ζ be any element of $\star$, i.e. $\mu(\zeta) \subset w$. Then $\xi(\mu(\zeta)) \prec \xi(w)$. Note that $\xi(\mu(\zeta)) = \zeta$ by the theory (Oort) on the minimal p -divisible groups. Thus we have $\zeta \prec \xi(w)$.

From this claim it suffices to prove Prop. $\Rightarrow$ (2). The proof is by induction of w w.r.t $\subset$. If w is minimal, we have $\mu(\xi(w)) = \mu(w) = w$. Assume w is not minimal. We now assume Proposition, which is paraphrased as $\dim \mathcal{S}_w(\mathcal{M}) > 0$, where $\mathcal{M}$ is the moduli space (over k) of isogenies $H(\xi(w)) \rightarrow Y$. Choose an irreducible component $\mathcal{I}$ of $\mathcal{M}$ such that $\dim \mathcal{S}_w(\mathcal{I}) > 0$. It is known that $\mathcal{I}$ is projective and $\mathcal{S}_w(\mathcal{I})$ is quasi-affine. Take a point $\in \mathcal{I} \cap \partial \mathcal{S}_w(\mathcal{I})$. Let w' be the p -kernel type of the point. Clearly w' satisfies $w' \subsetneq w$ and $\xi(w') = \xi(w)$. By the hypothesis of induction we may assume $\mu(\xi(w')) \subset w'$; then $\mu(\xi(w)) = \mu(\xi(w')) \subset w' \subset w$.

Outline of the proof of Proposition: By the existence of $\xi(w)$, there exists a p -divisible group X such that $X[p]$ is of type w and the Newton polygon of X is $\xi(w)$.

Step 1: We extract a simple first-slope part X_1 from X :

$$0 \longrightarrow X'_0 \longrightarrow X \xrightarrow{f_0} X_1 \longrightarrow 0 \quad (\text{exact})$$

Then the first-slope theory shows that $X_1 \simeq H_{n,m}$.

Take these p -kernels:

$$0 \longrightarrow X'_0[p] \longrightarrow X[p] \xrightarrow{\phi_0} X_1[p] \longrightarrow 0 \quad (\text{exact})$$

Step 2: Find a generic part S of the hom-space $\text{Hom}(X[p], X_1[p])$ whose $\phi : X[p]_S \rightarrow X_1[p]_S$ makes

$$0 \longrightarrow G \longrightarrow X[p]_S \xrightarrow{\phi} X_1[p]_S \longrightarrow 0 \quad (\text{exact})$$

so that G is a geometrically-constant BT_1 over S .

Step 3: We extend this to a complex over S' (finite/ S):

$$0 \longrightarrow X'_{S'} \longrightarrow \mathcal{X} \xrightarrow{f} X_{1,S'} \longrightarrow 0 \quad (\text{exact}),$$

so that we have a non-constant family $\mathcal{X} \rightarrow S'$.

5 Expectations

Note $\mathcal{W}_\xi^0$ has complicated singularities in general. We have a natural decomposition

$$\mathcal{W}_\xi^0 = \coprod \mathcal{W}_\xi^0 \cap \mathcal{S}_w.$$

Open problem: Can $\mathcal{W}_\xi^0 \cap \mathcal{S}_w$ be beautifully described? (regular?)

Note $\mathcal{W}_\xi^0 \cap \mathcal{S}_w$ is regular for $g \leq 3$. At least we expect:

Expectation 5.1. $\mathcal{W}_{\xi(w)}^0 \cap \mathcal{S}_w$ would be beautifully described.

Here $\xi(w)$ is the generic Newton polygon of $\mathcal{S}_w$. We have investigated the case $\xi(w) = \sigma$, i.e., $\mathcal{S}_w \subset \mathcal{W}_\sigma$:

Theorem 5.2 (H., to appear in J. Algebraic Geom.). *For any $w' \in \overline{W}_c'$ with $c \leq [g/2]$, there exists a finite surjective morphism*

$$G(\mathbb{Q}) \backslash X(w') \times G(\mathbb{A}_f)/K \rightarrow \bigcup_{\mathfrak{r}(w)=w'} \mathcal{S}_w,$$

which is bijective on geometric points.

Here $X(w')$ is the (generalized) Deligne-Lusztig variety:

$$\{P \in \mathrm{Sp}_{2c}/P_0 \mid {}^h P = P_0, {}^h \mathrm{Fr}(P) = {}^{w'} P_0 \text{ for } \exists h \in \mathrm{Sp}_{2c}\},$$

and G is a certain quaternion unitary group over $\mathbb{Q}$.

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