<table>
<thead>
<tr>
<th>Title</th>
<th>The supremum of Newton polygons of p-divisible groups with a given p-kernel type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Harashita, Shushi</td>
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<td>Departmental Bulletin Paper</td>
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Kyoto University
The supremum of Newton polygons of $p$-divisible groups with a given $p$-kernel type

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Abstract

We show that there exists the supremum of Newton polygons of $p$-divisible groups with a given $p$-kernel type, and provide an algorithm determining it. This is an unpolarized analogue of Oort conjecture related to determining the generic Newton polygon of each Ekedahl-Oort stratum in the moduli space of principally polarized abelian varieties.

1 Introduction

Let $\mathcal{A}_g$ be the moduli space (over $\mathbb{Z}$) of principally polarized abelian varieties of dimension $g$. It is well-known that

$$\mathcal{A}_g(\mathbb{C}) = \text{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H},$$

where $\mathbb{H}$ is the Siegel upper half space

$$\mathbb{H} = \{ Z \in M_g(\mathbb{C}) \mid Z = Z^\dagger, \text{Im } Z > 0 \}.$$ 

From now on we write $\mathcal{A}_g := \mathcal{A}_g \otimes \mathbb{F}_p$. Here is an expectation (so-called a paving of $\mathcal{A}_g$):

1. There exists a natural decomposition of $\mathcal{A}_g$ into finitely many locally closed subschemes:

   $$\mathcal{A}_g = \coprod_{\nu} \mathcal{T}_{\nu}.$$

2. Each $\mathcal{T}_{\nu}$ can be beautifully described.

Here this decomposition should be a decomposition by natural invariants of $p$-divisible groups of abelian varieties.
Let $S$ be a connected scheme. Let $p$ be a prime number. A $p$-divisible group over $S$ of height $h$ is an inductive system

$$X = \lim_{i \in \mathbb{N}} X_i,$$

of finite locally free group schemes $X_i$ of rank $p^i$ over $S$ such that

$$X_i = X_{i+1}[p],$$

where $G[N] := \text{Ker}(N : G \to G)$. For example

$$\mathbb{Q}_p/\mathbb{Z}_p, \quad \mathbb{G}_m[p^\infty], \quad A[p^\infty]$$

with an abelian scheme $A$ over $S$, where

$$G[p^\infty] = \lim_{i \in \mathbb{N}} G[p^i].$$

Let $k$ be an algebraically closed field of characteristic $p$. We have two invariants of a $p$-divisible group $X$ over $k$.

1. $\mathcal{N}(X) :=$ the isogeny class (= Newton polygon) of $X$, Dieudonné-Manin classification (1963);

2. $\mathcal{E}(X) :=$ the isomorphism class of $X[p]$, Kraft’s classification (1975).

We want to estimate $\mathcal{N}(X)$ from $\mathcal{E}(X)$.

**Today’s aim:** Let $w$ be any $p$-kernel type. We give a combinatorial algorithm determining the Newton polygon $\xi(w)$ satisfying

$$\forall X, \quad \mathcal{E}(X) = w \implies \mathcal{N}(X) < \xi(w),$$

$$\exists Y, \quad \mathcal{E}(Y) = w \quad \text{and} \quad \mathcal{N}(Y) = \xi(w).$$

The existence of the optimal upper bound $\xi(w)$ is non-trivial.

The (principally) polarized case - $\text{Sp}_{2g}$ (2007):

The problem obtained by replacing “$p$-divisible group” by “principally polarized $p$-divisible group”. We use the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties and the theory on stratifications on $\mathcal{A}_g$.

The unpolarized case - $\text{GL}_r$ (Today):

No natural moduli space!

Instead we treat families of $p$-divisible groups and families of $F$-zips, and consider stratifications on those.
Geometric meaning in the polarized case:

\[ \mathcal{A}_g = \prod_{\xi} \mathcal{W}_\xi^0 \] : Newton polygon stratification,

\[ \mathcal{A}_g = \prod_{w} \mathcal{S}_w \] : Ekedahl-Oort stratification,

\[ \mathcal{W}_\xi^0 := \{ A \in \mathcal{A}_g | \mathcal{N}(A) = \xi \}, \]

\[ \mathcal{S}_w := \{ A \in \mathcal{A}_g | \mathcal{E}(A) = w \}. \]

Open problem:

(1) When \( \mathcal{W}_\xi^0 \cap \mathcal{S}_w = \emptyset \)?

(2) Can \( \mathcal{W}_\xi^0 \cap \mathcal{S}_w \) be beautifully described?

Today’s aim in the pol. case \( \iff \) When \( \mathcal{S}_w \subset \overline{\mathcal{W}_\xi^0} \)?

\section{Preliminaries}

\subsection{The Dieudonné theory}

Let \( K \) be a perfect field. Let \( A_K \) denote the ring

\[ W(K)[\mathcal{F}, \mathcal{V}]/(\mathcal{F}a - a^\sigma \mathcal{F}, \mathcal{V}a^\sigma - a\mathcal{V}, \mathcal{FV} - p, \mathcal{VF} - p), \]

where \( \sigma \) is the Frobenius map \( W(K) \to W(K) \).

\textbf{Definition 2.1.} A Dieudonné module (DM) over \( K \) is a left \( A_K \)-module which is finitely generated as a \( W(K) \)-module.

\textbf{Theorem 2.2} (Dieudonné theory). \textit{There are categorical equivalences:}

\[ \mathbb{D} : \{ \text{p-divisible groups}/K \} \simeq \{ \text{DM}/K \text{ free as } W(K) \text{-mod.} \} \]

\[ \mathbb{D} : \{ \text{fin. } p\text{-group sch.}/K \} \simeq \{ \text{DM}/K \text{ of fin. length} \} \]

\subsection{Minimal p-divisible groups}

For a pair \( (m, n) \) of coprime non-negative integers, we define a \( p \)-divisible group \( H_{m,n} \) over \( \mathbb{F}_p \) by

\[ \mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p e_i \]

with \( \mathcal{F}e_i = e_{i+n}, \mathcal{V}e_i = e_{i+m} \) and \( e_{i+m+n} = pe_i \ (i \in \mathbb{Z}_{\geq 0}) \).

Let \( \xi \) be a Newton polygon \( \sum_{i=1}^t (m_i, n_i) \) (a formal sum).

\textbf{Definition 2.3.} A \textit{minimal } \( p\)-\textit{divisible group} of \( \xi \) is the \( p \)-divisible group

\[ H(\xi) = \bigoplus_{i=1}^{t} H_{m_i, n_i}. \]
2.3 Newton polygons

A Newton polygon $\xi = \sum_{i=1}^{t} (m_i, n_i)$ is regarded as a lower convex polygon with $(m_i + n_i)$ slopes $\lambda_i := m_i / (m_i + n_i)$ ($\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{t-1} \leq \lambda_t$).

$\zeta < \xi \iff \forall$ point of $\zeta$ is above or on $\xi$.

Let $X$ be a $p$-divisible group over $k = \overline{k}$. We write $\mathcal{N}(X) = \xi$ if $X$ is isogenous to $H(\xi)$.

**Theorem 2.4** (Dieudonné-Manin classification). We have a natural bijection:

$\mathcal{N} : \{p\text{-divisible groups over } k\}/\text{isog.} \simeq \{\text{Newton polygons}\}$.

We call $\xi$ symmetric if $\lambda_i + \lambda_{t+1-i} = 1$. Note $\mathcal{N}(A) := \mathcal{N}(A[p^\infty])$ for $A \in \mathcal{A}_g(k)$ is symmetric.

2.4 Final elements in the Weyl groups

Let $W_G$ denote the Weyl group of $G = \text{GL}_r$ or $\text{Sp}_{2g}$.

$W_{\text{GL}_r} = \text{Aut}\{1, \ldots, r\}$,

$W_{\text{Sp}_{2g}} = \{w \in W_{\text{GL}_{2g}} \mid w(i) + w(2g + 1 - i) = 2g + 1\}$.

We define a subset $J W_G$ of $W_G$ by

$J W_{\text{GL}_r} := \left\{ w \in W_{\text{GL}_r} \mid w^{-1}(1) < \cdots < w^{-1}(d), \begin{array}{c} w^{-1}(d + 1) < \cdots < w^{-1}(r) \end{array} \right\}$,

$J W_{\text{Sp}_{2g}} := \{ w \in W_{\text{Sp}_{2g}} \mid w^{-1}(1) < \cdots < w^{-1}(g) \}$,

where $J = \{s_1, \ldots, s_{r-1}\} \setminus \{s_d\}$ resp. $J = \{s_1, \ldots, s_g\} \setminus \{s_g\}$.

An element of $J W_G$ is called a final element of $W_G$.

A BT$_1$ over $S$ is a finite locally free group scheme $G$ over $S$ such that

$\text{Ker}(F : G \to G(p)) = \text{Im}(V : G(p) \to G)$,

$\text{Im}(F : G \to G(p)) = \text{Ker}(V : G(p) \to G)$.

Let $k$ be an algebraically closed field of characteristic $p$.

**Theorem 2.5** (Kraft, Oort, Moonen, Wedhorn).

$\{\text{BT}_1 \text{'s over } k \text{ of rank } p^r \text{ and dimension } d\}_{/\simeq} \simeq J W_{\text{GL}_r}$

$\{\text{polarized BT}_1 \text{'s over } k \text{ of rank } p^{2g}\}_{/\simeq} \simeq J W_{\text{Sp}_{2g}}$.

Note that $G$ over $k$ is a BT$_1$ if and only if $G \simeq X[p]$ for a $p$-divisible group $X$ over $k$. A polarization on $G$ is a non-degenerate alternating form $\mathbb{D}(G) \otimes_k \mathbb{D}(G) \to k$ satisfying $\langle FX, y \rangle = \langle x, V y \rangle^\sigma$ for all $x, y \in \mathbb{D}(G)$.
3 The polarized case

3.1 Stratifications on \( \mathcal{A}_g \)

Let \( \mathcal{A}_g \) be the moduli space of principally polarized abelian varieties of dimension \( g \) in characteristic \( p \).

\[
\mathcal{A}_g = \coprod_{\xi} \mathcal{W}_{\xi}^0 : \text{Newton polygon stratification,}
\]

\[
\mathcal{A}_g = \coprod_{w} \mathcal{S}_w : \text{Ekedahl-Oort stratification,}
\]

where we define

\[
\mathcal{W}_{\xi}^0 := \{ A \in \mathcal{A}_g | \mathcal{N}(A) = \xi \},
\]

\[
\mathcal{S}_w := \{ A \in \mathcal{A}_g | \mathcal{E}(A) = w \}.
\]

3.2 Oort’s conjecture

Conjecture 3.1 (Oort).

\[
\mathcal{W}_{\xi}^0 \cap \mathcal{S}_w \neq \emptyset \implies \mathcal{Z}_\xi \subseteq \overline{\mathcal{S}_w}.
\]

Here \( \mathcal{Z}_\xi \) is defined to be

\[
\mathcal{Z}_\xi = \{ A \in \mathcal{A}_g | A[p^\infty]_\Omega \simeq H(\xi)[\Omega] \text{ for some } \Omega = \overline{\Omega} \},
\]

which is shown to be a closed subset of \( \mathcal{W}_{\xi}^0 \). We call \( \mathcal{Z}_\xi \) the central stream of \( \xi \). Oort showed

\[
\mathcal{Z}_\xi = \{ A \in \mathcal{A}_g | A[p]_\Omega \simeq H(\xi)[p]_\Omega \text{ for some } \Omega = \overline{\Omega} \}
\]

\[
= \mathcal{S}_{\mu(\xi)},
\]

where \( \mu(\xi) \) is the \( p \)-kernel type \( \mathcal{E}(H(\xi)) \) of \( H(\xi) \).

3.3 Irreducibility of Ekedahl-Oort strata

The irreducibility of \( \mathcal{S}_w \) depends on whether \( \mathcal{S}_w \subset \mathcal{W}_\sigma \).

Theorem 3.2 (Ekedahl - van der Geer). \( \mathcal{S}_w \) is irreducible if \( \mathcal{S}_w \not\subset \mathcal{W}_\sigma \).

Theorem 3.3 (H., to appear in J. Alg. Geom.). \( \mathcal{S}_w \) is reducible for \( p \gg 0 \) if \( \mathcal{S}_w \subset \mathcal{W}_\sigma \).
Definition 3.4. The generic Newton polygon of $S_w$ is defined to be

$$\xi(w) = \text{Newton polygon of a (every) generic point of } S_w.$$ 

By Grothendieck-Katz, $\xi(w)$ is the optimal upper bound:

$$\forall X, \; E(X) = w \implies N(X) \prec \xi(w),$$

$$\exists Y, \; E(Y) = w \; \& \; N(Y) = \xi(w).$$

3.4 Results

Theorem 3.5 (H., to appear in Ann. Inst. Fourier). For any $w \in \mathcal{J}_{\text{Sp}_{2g}}$, we have

$$\xi(w) = \max_{\prec} \{ \xi \mid Z_\xi \subset \overline{S}_w \}.$$ 

This gives a combinatorial algorithm determining the generic Newton polygon $\xi(w)$ of $S_w$. Recall that $Z_\xi = S_{\mu(\xi)}$, where $\mu(\xi)$ is the $p$-kernel type of $H(\xi)$.

Theorem 3.6 (H., Asian J. Math. (2009)).

$$Z_\xi \subset \overline{Z_\xi} \iff \zeta \prec \xi.$$ 

Corollary 3.7. Oort’s conjecture is true: $W^0_\xi \cap S_w \neq \emptyset \implies Z_\zeta \subset \overline{S}_w$.

4 The unpolarized case

4.1 Main results

Theorem 4.1 (H.). Let $w \in \mathcal{J}_{\text{GL}}$. The optimal upper bound $\xi(w)$ exists, and

$$\xi(w) = \max_{\prec} \{ \xi \mid \mu(\xi) \subset w \}.$$ 

This gives a combinatorial algorithm determining $\xi(w)$. See below for what $\subset$ means. Again recall $\mu(\xi) = E(H(\xi))$.

Theorem 4.2 (H.). $\mu(\zeta) \subset \mu(\xi) \iff \zeta \prec \xi.$

Corollary 4.3 (The unpolarized analogue of Oort’s conjecture). If there exists a $p$-divisible group $X$ with Newton polygon $\zeta$ and $p$-kernel type $w$, then we have $\mu(\zeta) \subset w$.

Because $\zeta \prec \xi(w)$ and therefore $\mu(\zeta) \subset \mu(\xi(w)) \subset w$. 

-14-
4.2 $F$-zips and displays

Let $S$ be an $\mathbb{F}_p$-scheme. Let $\sigma$ be the absolute Frobenius on $S$. For any $\mathcal{O}_S$-module $M$ we write $M^{(p)} = \mathcal{O}_S \otimes_{\sigma, \mathcal{O}_S} M$.

**Definition 4.4** (Moonen-Wedhorn). An $F$-zip over $S$ is a quintuple $Z = (N, C, D, \varphi, \dot{\varphi})$ consisting of locally free $\mathcal{O}_S$-module $N$ and $\mathcal{O}_S$-submodules $C, D$ of $N$ which are locally direct summands of $N$, and isomorphisms $\varphi : (N/C)^{(p)} \rightarrow D$ and $\dot{\varphi} : C^{(p)} \rightarrow N/D$.

If $S = \text{Spec}(K)$ with a perfect field $K$, then $f_{\text{BT}1}$’s over $K$ give a correspondence $\{\text{BT}_1 \text{’s over } K\} \sim \{F\text{-zips over } K\}$ sending $G$ to $(\mathcal{D}(G), VN, FN, \mathcal{F}, \mathcal{V}^{-1})$.

From now on we write $W = W_{GL_r}$ and $J W = J W_{GL_r}$.

**Definition 4.5.** Let $w, w' \in J W$. We say $w \lhd w'$ if there is an $F$-zip over a valuation ring such that the special fiber is of type $w$ and the generic fiber is of type $w'$.

**Theorem 4.6** (Wedhorn). (1) $\lhd$ gives an ordering on $J W$.

(2) There exists a combinatorial algorithm determining whether $w \lhd w'$ for concretely given $w$ and $w'$.

One can show that

**Lemma 4.7.** Let $w, w' \in J W_{GL_r}$. If $w \lhd w'$, then we have $\xi(w) < \xi(w')$.

Let $R$ be a commutative ring. Let $F$ and $V$ be the Frobenius and Verschiebung on $W(R)$. Put $I_R = V W(R)$.

A display over $R$ is a quadruple $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ of

(i) $P$: a finitely generated projective $W(R)$-module;

(ii) $Q$ : a submodule of $P$ such that $\exists$ decomposition $P = L \oplus T$ such that $Q = L \oplus I_R T$;

(iii) $\mathcal{F} : P^{(p)} \rightarrow P$ and $\mathcal{V}^{-1} : Q^{(p)} \rightarrow P$: $W(R)$-linear maps.

**Theorem 4.8** (Zink). Assume $R$ is an excellent local ring or of finite type over a field of char. $p$. Then

$\{\text{nilpotent displays over } R\} \simeq \{\text{formal } p\text{-div. gp. over } R\}$.

An $F$-zip over $R$ is the mod $I_R$-reduction of a display over $R$. 

-15-
4.3 The existence of $\xi(w)$

In the polarized case, the existence of $\xi(w)$ follows from the irreducibility of Ekedahl-Oort strata. Instead we prove

**Lemma 4.9.** There exists an irreducible catalogue of $p$-divisible groups with a given $p^m$-kernel type: Let $m \in \mathbb{N}$, and let $u$ be a $p^m$-kernel type. There exists a $p$-divisible group $X$ over an irreducible scheme $S$ of finite type over $k$ such that

1. every geometric fiber $X_s$ is of $p^m$-kernel type $u$;
2. For any $p$-divisible group $X$ with $p^m$-kernel type $u$, there exists a geometric point $s \in S$ such that $X \simeq X_s$.

This (for $m = 1$) proves that the optimal upper bound $\xi(w)$ exists. Indeed the Newton polygon of the generic fiber of $X$ satisfies all the properties of $\xi(w)$.

**Proof.** Let $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ be a display over $k$, and $P = L \oplus T$ be a normal decomposition. Let

(a) $G := \text{GL}(P)$ the general linear group over $W(k)$;
(b) $H$: the paraholic subgroup of $G$ stabilizing $Q$;
(c) $\mathcal{D}_m$: the group scheme over $k$ representing the functor

$$\text{Alg}_k \rightarrow \text{Set} : R \mapsto G(W_m(R));$$

(d) $\mathcal{H}_m$: the group scheme over $k$ representing the functor

$$\text{Alg}_k \rightarrow \text{Set} : R \mapsto H(W_m(R)).$$

We have that $\mathcal{D}_m$ and $\mathcal{H}_m$ are connected smooth affine group schemes over $k$, see Vasiu [J. Alg. Geom. (2008)]. For any truncated Barsotti-Tate group of level $m$ ($\text{BT}_m$) with codim. $c$ and dim. $d$, its Dieudonné module is written as $(P/p^mP, g\mathcal{F}, \mathcal{V}g^{-1})$ for some $g \in \mathcal{D}_m$. Let

$$\text{BT}_m(k) = \{\text{BT}_m \text{ over } k \text{ of codim. } c \text{ and dim. } d\}/\simeq.$$

Vasiu introduced an action:

$$\mathbb{T}_m : \mathcal{H}_m \times_k \mathcal{D}_m \longrightarrow \mathcal{D}_m,$$

and showed that

$$\{\mathbb{T}_m\text{-orbits}\} \simeq \text{BT}_m(k).$$
Now we can construct an irreducible catalogue of $p$-divisible groups with $p^m$-kernel type $u$.

Choose an integer $N \geq m$ so that $X[p^N] \simeq Y[p^N]$ implies $X \simeq Y$ for any $p$-divisible groups $X$ and $Y$ over $k$. Let $\pi$ be the natural map $\mathcal{D}_N \to \mathcal{D}_m$, and let $\tau$ be a section of $\mathcal{D} \to \mathcal{D}_N$. Let $\mathbb{O}_u$ be the $\mathbb{T}_m$-orbit associated to $u$. Since $\mathcal{H}_m$ is irreducible, $\mathbb{O}_u$ is irreducible. Since $\pi$ is smooth with connected fibers, $\pi^{-1}(\mathbb{O}_u)$ is also irreducible. Let $S$ be the image of $\pi^{-1}(\mathbb{O}_u)$ by $\tau$. Then $S$ is irreducible and of finite type over $k$. By Zink’s display theory, we have a $p$-divisible group $\mathcal{X}$ over $S$. Clearly $\mathcal{X}$ satisfies the required properties.

4.4 Outline of the proof (1st slope theory and induction)

Let $w \in \mathbb{J}_{\text{GL}_r}$. Set $\nu_w(i) = \{a \leq i \mid w(a) > d\}$. We define a map

$$\Psi_w : \{1, \ldots, r\} \to \{1, \ldots, r\}$$

by $\Psi_w(i) = d + i$ if $w(i) = i$ and $\Psi_w(i) = \nu_w(i)$ otherwise. Let

$$\mathcal{D} = \text{Im} \Psi_w^k \text{ for } k \gg 0,$$
$$\mathcal{C} = \mathcal{D} \cap \{d + 1, \ldots, r\}.$$

**Theorem 4.10** (H., J. Pure Appl. Algebra (2009)). (1) The last slope of $\xi(w)$ is equal to $\rho(w) := \#\mathcal{C} / \#\mathcal{D}$.

(2) $\rho(w) = \max\{m/(m + n) \mid H_{m,n}[p] \hookrightarrow G_w\}.$

The first slope $\lambda(w)$ of $\xi(w)$ is equal to $1 - \rho(w')$.

$$\lambda(w) = \min\{m/(m + n) \mid G_w \hookrightarrow H_{m,n}[p]\}.$$


To show the main theorem, it suffices to show

**Proposition 4.11.** Assume that $w$ is not minimal. Then there exists a non-constant family of isogenies of $p$-divisible groups

$$H(\xi(w))_S \longrightarrow \mathcal{X}$$

over $S$ such that the isomorphism type of $\mathcal{X}_s[p]$ is $w$ for every geometric point $s$ of $S$.

The main theorem follows from this proposition:

**Proof of Prop.** $\Rightarrow$ the main theorem. We first claim that the main theorem

$$\xi(w) = \max\{\zeta \mid \mu(\zeta) \subset w\}$$  (1)
is equivalent to
\[ \mu(\xi(w)) \subset w. \] (2)

Obviously (1) implies (2). Conversely suppose (2). Put \( \star = \{ \zeta \mid \mu(\zeta) \subset w \} \). Clearly (2) says \( \xi(w) \in \star \). Let \( \zeta \) be any element of \( \star \), i.e., \( \mu(\zeta) \subset w \). Then \( \xi(\mu(\zeta)) \prec \xi(w) \). Note that \( \xi(\mu(\zeta)) = \zeta \) by the theory (Oort) on the minimal \( p \)-divisible groups. Thus we have \( \zeta \prec \xi(w) \).

From this claim it suffices to prove Prop. \( \Rightarrow \) (2). The proof is by induction of \( w \) w.r.t. \( \frac{1}{2}w \). If \( w \) is minimal, we have \( \mu(\xi(w)) = \mu(w) = w \). Assume \( w \) is not minimal. We now assume Proposition, which is paraphrased as \( \dim S_w(M) > 0 \), where \( M \) is the moduli space (over \( k \)) of isogenies \( H(\xi(w)) \to Y \). Choose an irreducible component \( \mathcal{I} \) of \( M \) such that \( \dim S_w(\mathcal{I}) > 0 \). It is known that \( \mathcal{I} \) is projective and \( S_w(\mathcal{I}) \) is quasi-affine. Take a point \( \in \mathcal{I} \cap \partial S_w(\mathcal{I}) \). Let \( w' \) be the \( p \)-kernel type of the point. Clearly \( w' \) satisfies \( \xi(w') \subset w' \). Then \( \xi(\mu(\zeta(\mu(\zeta)))) = \zeta \) by the theory (Oort) on the minimal \( p \)-divisible groups. Thus we have \( \zeta \prec \xi(w') \).

Outline of the proof of Proposition: By the existence of \( \xi(w) \), there exists a \( p \)-divisible group \( X \) such that \( X[p] \) is of type \( w \) and the Newton polygon of \( X \) is \( \xi(w) \).

Step 1: We extract a simple first-slope part \( X_1 \) from \( X \):

\[ \begin{array}{cccc}
0 & \longrightarrow & X'_0 & \longrightarrow & X & \longrightarrow & X_1 & \longrightarrow & 0 \\
& & \phi_0 & & & \longrightarrow & & & \\
\end{array} \]

Then the first-slope theory shows that \( X_1 \simeq H_{n,m} \).

Take these \( p \)-kernels:

\[ \begin{array}{cccc}
0 & \longrightarrow & X'_0[p] & \longrightarrow & X[p] & \longrightarrow & X_1[p] & \longrightarrow & 0 \\
& & \phi_0 & & & \longrightarrow & & & \\
\end{array} \] (exact)

Step 2: Find a generic part \( S \) of the hom-space \( \Hom(X[p], X_1[p]) \) whose \( \phi : X[p]_S \to X_1[p]_S \) makes

\[ \begin{array}{cccc}
0 & \longrightarrow & G & \longrightarrow & X[p]_S & \longrightarrow & X_1[p]_S & \longrightarrow & 0 \\
& & \phi & & \longrightarrow & & & & \\
\end{array} \] (exact)

so that \( G \) is a geometrically-constant BT\(_1 \) over \( S \).

Step 3: We extend this to a complex over \( S' \) (finite/\( S' \)):

\[ \begin{array}{cccc}
0 & \longrightarrow & X'_0[S'] & \longrightarrow & X & \longrightarrow & X_1[S'] & \longrightarrow & 0 \\
& & \phi & & \longrightarrow & & & & \\
\end{array} \] (exact),

so that we have a non-constant family \( X \to S' \).

5 Expectations

Note \( \mathcal{W}_\xi^0 \) has complicated singularities in general. We have a natural decomposition

\[ \mathcal{W}_\xi^0 = \coprod \mathcal{W}_\xi^0 \cap S_w. \]
Open problem: Can $\mathcal{W}_\xi^0 \cap S_w$ be beautifully described? (regular?)

Note $\mathcal{W}_\xi^0 \cap S_w$ is regular for $g \leq 3$. At least we expect:

**Expectation 5.1.** $\mathcal{W}_\xi^0 \cap S_w$ would be beautifully described.

Here $\xi(w)$ is the generic Newton polygon of $S_w$. We have investigated the case $\xi(w) = \sigma$, i.e., $S_w \subset \mathcal{W}_\sigma$:

**Theorem 5.2** (H., to appear in J. Algebraic Geom.). For any $w' \in \overline{W}_e$ with $c \leq [g/2]$, there exists a finite surjective morphism

$$G(\mathbb{Q}) \backslash X(w') \times G(\mathbb{A}_f)/K \to \bigcup_{t(w) = w'} S_w,$$

which is bijective on geometric points.

Here $X(w')$ is the (generalized) Deligne-Lusztig variety:

$$\{ P \in \text{Sp}_{2c} / P_0 \mid {}^h P = P_0, {}^h \text{Fr}(P) = w' P_0 \text{ for } \exists h \in \text{Sp}_{2c} \},$$

and $G$ is a certain quaternion unitary group over $\mathbb{Q}$.

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