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The supremum of Newton polygons of p-divisible groups with a given p-kernel type

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Abstract

We show that there exists the supremum of Newton polygons of p-divisible groups with a given p-kernel type, and provide an algorithm determining it. This is an unpolarized analogue of Oort conjecture related to determining the generic Newton polygon of each Ekedahl-Oort stratum in the moduli space of principally polarized abelian varieties.

1 Introduction

Let $\mathcal{A}_g$ be the moduli space (over $\mathbb{Z}$) of principally polarized abelian varieties of dimension g. It is well-known that

$$\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H},$$

where $\mathbb{H}$ is the Siegel upper half space

$$\mathbb{H} = \{Z \in \mathrm{M}_g(\mathbb{C}) \mid Z = {}^tZ, \mathrm{Im}\, Z > 0\}.$$

From now on we write $\mathcal{A}_g := \mathcal{A}_g \otimes \mathbb{F}_p$. Here is an expectation (so-called a paving of $\mathcal{A}_g$):

(1) There exists a natural decomposition of $\mathcal{A}_g$ into nitely many locally closed subschemes:
$$\mathcal{A}_g = \coprod_\nu \mathcal{T}_\nu$$

(2) Each $\mathcal{T}_\nu$ can be beautifully described.

Here this decomposition should be a decomposition by natural invariants of p-divisible groups of abelian varieties.

Let S be a connected scheme. Let p be a prime number. A *p-divisible group over S* of height h is an inductive system

$$X = \varinjlim_{i \in \mathbb{N}} X_i, \qquad X_i \subset X_{i+1}$$

of nite locally free group schemes X_i of rank p^{ih} over S such that

$$X_i = X_{i+1}[p^i],$$

where $G[N] := \mathrm{Ker}(N : G \to G)$. For example

$$\mathbb{Q}_p/\mathbb{Z}_p, \quad \mathbb{G}_m[p^\infty], \quad A[p^\infty]$$

with an abelian scheme A over S, where

$$G[p^\infty] = \varinjlim_{i \in \mathbb{N}} G[p^i].$$

Let k be an algebraically closed eld of characteristic p. We have two invariants of a p-divisible group X over k.

(1) $\mathcal{N}(X)$:= the isogeny class (= Newton polygon) of X,
Dieudonné-Manin classi cation (1963);

(2) $\mathcal{E}(X)$:= the isomorphism class of $X[p]$,
Kraft's classi cation (1975).

We want to estimate $\mathcal{N}(X)$ from $\mathcal{E}(X)$.

Today's aim: Let w be any p-kernel type. We give a combinatorial algorithm determining the Newton polygon $\xi(w)$ satisfying

$$\begin{aligned} \forall X, \quad \mathcal{E}(X) = w \quad &\Longrightarrow \quad \mathcal{N}(X) \prec \xi(w), \\ \exists Y, \quad \mathcal{E}(Y) = w \quad &\text{and} \quad \mathcal{N}(Y) = \xi(w). \end{aligned}$$

The existence of the optimal upper bound $\xi(w)$ is non-trivial.
The (principally) polarized case - Sp_{2g} (2007):

The problem obtained by replacing "p-divisible group" by "principally polarized p-divisible group". We use the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties and the theory on strati cations on $\mathcal{A}_g$.

The unpolarized case - GL_r (Today):

No natural moduli space!

Instead we treat families of p-divisible groups and families of F-zips, and consider strati cations on those.

Geometric meaning in the polarized case:

$$\mathcal{A}_g = \coprod_{\xi} \mathcal{W}_{\xi}^0 \quad : \text{Newton polygon strati cation,}$$

$$\mathcal{A}_g = \coprod_{w} \mathcal{S}_w \quad : \text{Ekedahl-Oort strati cation,}$$

$$\begin{aligned} \mathcal{W}_{\xi}^0 &:= \{A \in \mathcal{A}_g \mid \mathcal{N}(A) = \xi\}, \\ \mathcal{S}_w &:= \{A \in \mathcal{A}_g \mid \mathcal{E}(A) = w\}. \end{aligned}$$

Open problem:

(1) When $\mathcal{W}_{\xi}^0 \cap \mathcal{S}_w = \emptyset$?

(2) Can $\mathcal{W}_{\xi}^0 \cap \mathcal{S}_w$ be beautifully described?

Today's aim in the pol. case $\iff$ When $\mathcal{S}_w \subset \overline{\mathcal{W}_{\xi}^0}$?

2 Preliminaries

2.1 The Dieudonné theory

Let K be a perfect eld. Let A_K denote the ring

$$W(K)[\mathcal{F}, \mathcal{V}]/(\mathcal{F}a - a^{\sigma}\mathcal{F}, \mathcal{V}a^{\sigma} - a\mathcal{V}, \mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p),$$

where σ is the Frobenius map $W(K) \to W(K)$.

Definition 2.1. A Dieudonné module (DM) over K is a left A_K-module which is nitely generated as a $W(K)$-module.

Theorem 2.2 (Dieudonné theory)**.** *There are categorical equivalences:*

$$\mathbb{D}: \ \{p\text{-}divisible\ groups/K\} \simeq \{DM/K\ free\ as\ W(K)\text{-}mod.\}$$

$$\mathbb{D}: \ \{fin.\ p\text{-}group\ sch./K\} \simeq \{DM/K\ \ of\ fin.\ length\}$$

2.2 Minimal p-divisible groups

For a pair (m, n) of coprime non-negative integers, we de ne a p-divisible group $H_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p e_i$$

with $\mathcal{F}e_i = e_{i+n}$, $\mathcal{V}e_i = e_{i+m}$ and $e_{i+m+n} = pe_i$ $(i \in \mathbb{Z}_{\geq 0})$.

Let ξ be a Newton polygon $\sum_{i=1}^{t}(m_i, n_i)$ (a formal sum).

Definition 2.3. A *minimal p-divisible group* of ξ is the p-divisible group

$$H(\xi) = \bigoplus_{i=1}^{t} H_{m_i,n_i}.$$

2.3 Newton polygons

A Newton polygon $\xi = \sum_{i=1}^{t}(m_i, n_i)$ is regarded as a lower convex polygon with $(m_i + n_i)$ slopes $\lambda_i := m_i/(m_i + n_i)$ $(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{t-1} \leq \lambda_t)$.

$$\zeta \prec \xi \Longleftrightarrow \forall \text{point of } \zeta \text{ is above or on } \xi.$$

Let X be a p-divisible group over $k = \overline{k}$. We write $\mathcal{N}(X) = \xi$ if X is isogenous to $H(\xi)$.

Theorem 2.4 (Dieudonné-Manin classi cation)**.** *We have a natural bijection:*

$$\mathcal{N} : \{p\text{-divisible groups over } k\}/\text{isog.} \simeq \{\text{Newton polygons}\}.$$

We call ξ symmetric if $\lambda_i + \lambda_{t+1-i} = 1$. Note $\mathcal{N}(A) := \mathcal{N}(A[p^\infty])$ for $A \in \mathcal{A}_g(k)$ is symmetric.

2.4 Final elements in the Weyl groups

Let $\mathrm{W_G}$ denote the Weyl group of $\mathrm{G} = \mathrm{GL}_r$ or Sp_{2g}.

$$\begin{aligned} \mathrm{W}_{\mathrm{GL}_r} &= \mathrm{Aut}\{1, \dots, r\}, \\ \mathrm{W}_{\mathrm{Sp}_{2g}} &= \{w \in \mathrm{W}_{\mathrm{GL}_{2g}} \mid w(i) + w(2g + 1 - i) = 2g + 1\}. \end{aligned}$$

We de ne a subset ${}^{\mathrm{J}}\mathrm{W_G}$ of $\mathrm{W_G}$ by

$$\begin{aligned} {}^{\mathrm{J}}\mathrm{W}_{\mathrm{GL}_r} &:= \left\{ w \in \mathrm{W}_{\mathrm{GL}_r} \;\middle|\; \begin{array}{l} w^{-1}(1) < \cdots < w^{-1}(d), \\ w^{-1}(d+1) < \cdots < w^{-1}(r) \end{array} \right\}, \\ {}^{\mathrm{J}}\mathrm{W}_{\mathrm{Sp}_{2g}} &:= \{w \in \mathrm{W}_{\mathrm{Sp}_{2g}} \mid w^{-1}(1) < \cdots < w^{-1}(g)\}, \end{aligned}$$

where $\mathrm{J} = \{s_1, \dots, s_{r-1}\} \setminus \{s_d\}$ resp. $\mathrm{J} = \{s_1, \dots, s_g\} \setminus \{s_g\}$.

An element of ${}^{\mathrm{J}}\mathrm{W_G}$ is called a nal element of $\mathrm{W_G}$.

A BT_1 over S is a nite locally free group scheme G over S such that

$$\begin{aligned} \mathrm{Ker}(F : G \to G^{(p)}) &= \mathrm{Im}(V : G^{(p)} \to G), \\ \mathrm{Im}(F : G \to G^{(p)}) &= \mathrm{Ker}(V : G^{(p)} \to G). \end{aligned}$$

Let k be an algebraically closed eld of characteristic p.

Theorem 2.5 (Kraft, Oort, Moonen, Wedhorn)**.**

$$\{\mathrm{BT}_1\text{'s over } k \text{ of rank } p^r \text{ and dimension } d\}_{/\simeq} \simeq {}^{\mathrm{J}}\mathrm{W}_{\mathrm{GL}_r}$$

$$\{\text{polarized } \mathrm{BT}_1\text{'s over } k \text{ of rank } p^{2g}\}_{/\simeq} \simeq {}^{\mathrm{J}}\mathrm{W}_{\mathrm{Sp}_{2g}}.$$

Note that G over k is a BT_1 if and only if $G \simeq X[p]$ for a p-divisible group X over k. A polarization on G is a non-degenerate alternating form $\mathbb{D}(G) \otimes_k \mathbb{D}(G) \to k$ satisfying $\langle \mathcal{F}x, y \rangle = \langle x, \mathcal{V}y \rangle^\sigma$ for all $x, y \in \mathbb{D}(G)$.

3 The polarized case

3.1 Stratifications on $\mathcal{A}_g$

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g in characteristic p.

$$\begin{aligned} \mathcal{A}_g &= \coprod_{\xi} \mathcal{W}^0_\xi && : \text{Newton polygon strati cation},\\ \mathcal{A}_g &= \coprod_{w} \mathcal{S}_w && : \text{Ekedahl-Oort strati cation}, \end{aligned}$$

where we de ne

$$\begin{aligned} \mathcal{W}^0_\xi &:= \{A \in \mathcal{A}_g \mid \mathcal{N}(A) = \xi\},\\ \mathcal{S}_w &:= \{A \in \mathcal{A}_g \mid \mathcal{E}(A) = w\}. \end{aligned}$$

3.2 Oort's conjecture

Conjecture 3.1 (Oort)**.**

$$\mathcal{W}^0_\xi \cap \mathcal{S}_w \neq \emptyset \quad \Rightarrow \quad \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}$$

Here $\mathcal{Z}_\xi$ is de ned to be

$$\mathcal{Z}_\xi = \{A \in \mathcal{A}_g \mid A[p^\infty]_\Omega \simeq H(\xi)_\Omega \text{ for some } \Omega = \overline{\Omega}\},$$

which is shown to be a closed subset of $\mathcal{W}^0_\xi$. We call $\mathcal{Z}_\xi$ the central stream of ξ. Oort showed

$$\begin{aligned} \mathcal{Z}_\xi &= \{A \in \mathcal{A}_g \mid A[p]_\Omega \simeq H(\xi)[p]_\Omega \text{ for some } \Omega = \overline{\Omega}\}\\ &= \mathcal{S}_{\mu(\xi)}, \end{aligned}$$

where $\mu(\xi)$ is the p-kernel type $\mathcal{E}(H(\xi))$ of $H(\xi)$.

3.3 Irreducibility of Ekedahl-Oort strata

The irreducibility of $\mathcal{S}_w$ depends on whether $\mathcal{S}_w \subset \mathcal{W}_\sigma$.

Theorem 3.2 (Ekedahl - van der Geer)**.** *$\mathcal{S}_w$ is irreducible if $\mathcal{S}_w \not\subset \mathcal{W}_\sigma$.*

Theorem 3.3 (H., to appear in J. Alg. Geom.)**.** *$\mathcal{S}_w$ is reducible for $p \gg 0$ if $\mathcal{S}_w \subset \mathcal{W}_\sigma$.*

Definition 3.4. The *generic Newton polygon* of $\mathcal{S}_w$ is de ned to be

$$\xi(w) = \text{Newton polygon of a (every) generic point of } \mathcal{S}_w.$$

By Grothendieck-Katz, $\xi(w)$ is the optimal upper bound:

$$\begin{aligned} \forall X, \quad \mathcal{E}(X) = w \quad &\Longrightarrow \quad \mathcal{N}(X) \prec \xi(w), \\ \exists Y, \quad \mathcal{E}(Y) = w \quad &\& \quad \mathcal{N}(Y) = \xi(w). \end{aligned}$$

3.4 Results

Theorem 3.5 (H., to appear in Ann. Inst. Fourier)**.** *For any $w \in {}^{\mathrm{J}}\mathrm{W}_{\mathrm{Sp}_{2g}}$, we have*

$$\xi(w) = \max_{\prec}\{\xi \mid \mathcal{Z}_\xi \subset \overline{\mathcal{S}_w}\}.$$

This gives a combinatorial algorithm determining the generic Newton polygon $\xi(w)$ of $\mathcal{S}_w$. Recall that $\mathcal{Z}_\xi = \mathcal{S}_{\mu(\xi)}$, where $\mu(\zeta)$ is the p-kernel type of $H(\xi)$.

Theorem 3.6 (H., Asian J. Math. (2009))**.**

$$\mathcal{Z}_\zeta \subset \overline{\mathcal{Z}_\xi} \quad \Leftrightarrow \quad \zeta \prec \xi.$$

Corollary 3.7. *Oort's conjecture is true: $\mathcal{W}^0_\zeta \cap \mathcal{S}_w \neq \emptyset \Rightarrow \mathcal{Z}_\zeta \subset \overline{\mathcal{S}_w}$.*

4 The unpolarized case

4.1 Main results

Theorem 4.1 (H.)**.** *Let $w \in {}^{\mathrm{J}}\mathrm{W}_{\mathrm{GL}_r}$. The optimal upper bound $\xi(w)$ exists, and*

$$\xi(w) = \max_{\prec}\{\xi \mid \mu(\xi) \subset w\}.$$

This gives a combinatorial algorithm determining $\xi(w)$. See below for what $\subset$ means. Again recall $\mu(\xi) = \mathcal{E}(H(\xi))$.

Theorem 4.2 (H.)**.** $\mu(\zeta) \subset \mu(\xi) \quad \Leftrightarrow \quad \zeta \prec \xi.$

Corollary 4.3 (The unpolarized analogue of Oort's conjecture)**.** *If there exists a p-divisible group X with Newton polygon ζ and p-kernel type w, then we have $\mu(\zeta) \subset w$.*

Because $\zeta \prec \xi(w)$ and therefore $\mu(\zeta) \subset \mu(\xi(w)) \subset w$.

4.2 F-zips and displays

Let S be an $\mathbb{F}_p$-scheme. Let σ be the absolute Frobenius on S. For any $\mathcal{O}_S$-module M we write $M^{(p)} = \mathcal{O}_S \otimes_{\sigma,\mathcal{O}_S} M$.

Definition 4.4 (Moonen-Wedhorn)**.** An F-zip over S is a quintuple $Z = (N, C, D, \varphi, \dot{\varphi})$ consisting of locally free $\mathcal{O}_S$-module N and $\mathcal{O}_S$-submodules C, D of N which are locally direct summands of N, and isomorphisms $\varphi : (N/C)^{(p)} \to D$ and $\dot{\varphi} : C^{(p)} \to N/D$.

If $S = \operatorname{Spec}(K)$ with a perfect eld K, then

$$\{\mathrm{BT}_1\text{'s over } K\} \xrightarrow{\sim} \{F\text{-zips over } K\}$$

sending G to $(\mathbb{D}(G), \mathcal{V}N, \mathcal{F}N, \mathcal{F}, \mathcal{V}^{-1})$.

From now on we write $\mathrm{W} = \mathrm{W}_{\mathrm{GL}_r}$ and ${}^{\mathrm{J}}\mathrm{W} = {}^{\mathrm{J}}\mathrm{W}_{\mathrm{GL}_r}$.

Definition 4.5. Let $w, w' \in {}^{\mathrm{J}}\mathrm{W}$. We say $w \subset w'$ if there is an F-zip over a valuation ring such that the special ber is of type w and the generic ber is of type w'.

Theorem 4.6 (Wedhorn)**.** (1) *$\subset$ gives an ordering on ${}^{\mathrm{J}}\mathrm{W}$.*

(2) *There exists a combinatorial algorithm determining whether $w \subset w'$ for concretely given w and w'.*

One can show that

Lemma 4.7. *Let $w, w' \in {}^{\mathrm{J}}\mathrm{W}_{\mathrm{GL}_r}$. If $w \subset w'$, then we have $\xi(w) \prec \xi(w')$.*

Let R be a commutative ring. Let ^{F} and ^{V} be the Frobenius and Verschiebung on $W(R)$. Put $I_R = {}^V W(R)$.

A display over R is a quadruple $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ of

(i) P: a nitely generated projective $W(R)$-module;

(ii) Q : a submodule of P such that $\exists$ decomposition $P = L \oplus T$ such that $Q = L \oplus I_R T$;

(iii) $\mathcal{F} : P^{(p)} \to P$ and $\mathcal{V}^{-1} : Q^{(p)} \twoheadrightarrow P$: $W(R)$-linear maps.

Theorem 4.8 (Zink)**.** *Assume R is an excellent local ring or of finite type over a field of char. p. Then*

$$\{\text{nilpotent displays over } R\} \simeq \{\text{formal } p\text{-div. gp. over } R\}.$$

An F-zip over R is the mod I_R-reduction of a display over R.

4.3 The existence of $\xi(w)$

In the polarized case, the existence of $\xi(w)$ follows from the irreducibility of Ekedahl-Oort strata. Instead we prove

Lemma 4.9. *There exists an irreducible catalogue of p-divisible groups with a given p^m-kernel type: Let $m \in \mathbb{N}$, and let u be a p^m-kernel type. There exists a p-divisible group $\mathcal{X}$ over an irreducible scheme S of finite type over k such that*

(1) *every geometric fiber $\mathcal{X}_s$ is of p^m-kernel type u;*

(2) *For any p-divisible group X with p^m-kernel type u, there exists a geometric point $s \in S$ such that $X \simeq \mathcal{X}_s$.*

This (for $m = 1$) proves that the optimal upper bound $\xi(w)$ exists. Indeed the Newton polygon of the generic ber of $\mathcal{X}$ satis es all the properties of $\xi(w)$.

Proof. Let $(P, Q, \mathcal{F}, \mathcal{V}^{-1})$ be a display over k, and $P = L \oplus T$ be a normal decomposition. Let

(a) $\mathrm{G} := \mathrm{GL}(P)$ the general linear group over $W(k)$;

(b) H: the paraholic subgroup of G stabilizing Q;

(c) $\mathcal{D}_m$: the group scheme over k representing the functor

$$\mathrm{Alg}_k \to \mathrm{Set} : \quad R \mapsto \mathrm{G}(W_m(R));$$

(d) $\mathcal{H}_m$: the group scheme over k representing the functor

$$\mathrm{Alg}_k \to \mathrm{Set} : \quad R \mapsto \mathrm{H}(W_m(R)).$$

We have that $\mathcal{D}_m$ and $\mathcal{H}_m$ are connected smooth affine group schemes over k, see Vasiu [J. Alg. Geom. (2008)]. For any truncated Barsotti-Tate group of level m (BT_m) with codim. c and dim. d, its Dieudonné module is written as $(P/p^m P, g\mathcal{F}, \mathcal{V}g^{-1})$ for some $g \in \mathcal{D}_m$. Let

$$\mathbf{BT}_m(k) = \{\mathrm{BT}_m \text{ over } k \text{ of codim. } c \text{ and dim. } d\}/\simeq .$$

Vasiu introduced an action:

$$\mathbb{T}_m : \quad \mathcal{H}_m \times_k \mathcal{D}_m \longrightarrow \mathcal{D}_m,$$

and showed that

$$\{\mathbb{T}_m\text{-orbits}\} \simeq \mathbf{BT}_m(k).$$

Now we can construct an irreducible catalogue of p-divisible groups with p^m-kernel type u.

Choose an integer $N \geq m$ so that $X[p^N] \simeq Y[p^N]$ implies $X \simeq Y$ for any p-divisible groups X and Y over k. Let π be the natural map $\mathcal{D}_N \to \mathcal{D}_m$, and let τ be a section of $\mathcal{D} \to \mathcal{D}_N$. Let $\mathbb{O}_u$ be the $\mathbb{T}_m$-orbit associated to u. Since $\mathcal{H}_m$ is irreducible, $\mathbb{O}_u$ is irreducible. Since π is smooth with connected bers, $\pi^{-1}(\mathbb{O}_u)$ is also irreducible. Let S be the image of $\pi^{-1}(\mathbb{O}_u)$ by τ. Then S is irreducible and of nite type over k. By Zink's display theory, we have a p-divisible group $\mathcal{X}$ over S. Clearly $\mathcal{X}$ satis es the required properties.

4.4 Outline of the proof (1st slope theory and induction)

Let $w \in {}^{\mathrm{J}}\mathrm{W}_{\mathrm{GL}_r}$. Set $\nu_w(i) = \sharp\{a \leq i \mid w(a) > d\}$. We de ne a map

$$\Psi_w : \{1, \dots, r\} \to \{1, \dots, r\}$$

by $\Psi_w(i) = d + i$ if $w(i) = i$ and $\Psi_w(i) = \nu_w(i)$ otherwise. Let

$$\begin{aligned} \mathcal{D} &= \operatorname{Im} \Psi_w^k \quad \text{for } k \gg 0, \\ \mathcal{C} &= \mathcal{D} \cap \{d+1, \dots, r\}. \end{aligned}$$

Theorem 4.10 (H., J. Pure Appl. Algebra (2009)). (1) *The last slope of* $\xi(w)$ *is equal to* $\rho(w) := \sharp\mathcal{C}/\sharp\mathcal{D}$.

(2) $\rho(w) = \max\{m/(m+n) \mid H_{m,n}[p] \overset{\exists}{\hookrightarrow} G_w\}$.

The rst slope $\lambda(w)$ of $\xi(w)$ is equal to $1 - \rho(w^\vee)$.

$\lambda(w) = \min\{m/(m+n) \mid G_w \overset{\exists}{\twoheadrightarrow} H_{m,n}[p]\}$.

For the polarized case, see H., J. Alg. Geom. (2007).

To show the main theorem, it suffices to show

Proposition 4.11. *Assume that* w *is not minimal. Then there exists a non-constant family of isogenies of* p*-divisible groups*

$$H(\xi(w))_S \longrightarrow \mathcal{X}$$

over S *such that the isomorphism type of* $\mathcal{X}_s[p]$ *is* w *for every geometric point* s *of* S.

The main theorem follows from this proposition:

Proof of Prop. $\Rightarrow$ *the main theorem.* We rst claim that the main theorem

$$\xi(w) = \max\{\zeta \mid \mu(\zeta) \subset w\} \tag{1}$$

is equivalent to

$$\mu(\xi(w)) \subset w. \tag{2}$$

Obviously (1) implies (2). Conversely suppose (2). Put $\bigstar = \{\zeta \mid \mu(\zeta) \subset w\}$. Clearly (2) says $\xi(w) \in \bigstar$. Let ζ be any element of $\bigstar$, i.e, $\mu(\zeta) \subset w$. Then $\xi(\mu(\zeta)) \prec \xi(w)$. Note that $\xi(\mu(\zeta)) = \zeta$ by the theory (Oort) on the minimal p-divisible groups. Thus we have $\zeta \prec \xi(w)$.

From this claim it suffices to prove Prop. $\Rightarrow$ (2). The proof is by induction of w w.r.t $\subset$. If w is minimal, we have $\mu(\xi(w)) = \mu(w) = w$. Assume w is not minimal. We now assume Proposition, which is paraphrased as $\dim \mathcal{S}_w(\mathcal{M}) > 0$, where $\mathcal{M}$ is the moduli space (over k) of isogenies $H(\xi(w)) \to Y$. Choose an irreducible component $\mathcal{I}$ of $\mathcal{M}$ such that $\dim \mathcal{S}_w(\mathcal{I}) > 0$. It is known that $\mathcal{I}$ is projective and $\mathcal{S}_w(\mathcal{I})$ is quasi-affine. Take a point $\in \mathcal{I} \cap \partial\mathcal{S}_w(\mathcal{I})$. Let w' be the p-kernel type of the point. Clearly w' satis es $w' \subsetneq w$ and $\xi(w') = \xi(w)$. By the hypothesis of induction we may assume $\mu(\xi(w')) \subset w'$; then $\mu(\xi(w)) = \mu(\xi(w')) \subset w' \subset w$.

Outline of the proof of Proposition: By the existence of $\xi(w)$, there exists a p-divisible group X such that $X[p]$ is of type w and the Newton polygon of X is $\xi(w)$.

Step 1: We extract a simple rst-slope part X_1 from X:

$$0 \longrightarrow X_0' \longrightarrow X \xrightarrow{f_0} X_1 \longrightarrow 0 \quad \text{(exact)}$$

Then the rst-slope theory shows that $X_1 \simeq H_{n,m}$.

Take these p-kernels:

$$0 \longrightarrow X_0'[p] \longrightarrow X[p] \xrightarrow{\phi_0} X_1[p] \longrightarrow 0 \quad \text{(exact)}$$

Step 2: Find a generic part S of the hom-space $\mathrm{Hom}(X[p], X_1[p])$ whose $\phi : X[p]_S \to X_1[p]_S$ makes

$$0 \longrightarrow G \longrightarrow X[p]_S \xrightarrow{\phi} X_1[p]_S \longrightarrow 0 \quad \text{(exact)}$$

so that G is a geometrically-constant BT_1 over S.

Step 3: We extend this to a complex over S' (nite/S):

$$0 \longrightarrow X_{S'}' \longrightarrow \mathcal{X} \xrightarrow{f} X_{1,S'} \longrightarrow 0 \quad \text{(exact)},$$

so that we have a non-constant family $\mathcal{X} \to S'$.

5 Expectations

Note $\mathcal{W}_\xi^0$ has complicated singularities in general. We have a natural decomposition

$$\mathcal{W}_\xi^0 = \coprod \mathcal{W}_\xi^0 \cap \mathcal{S}_w.$$

Open problem: Can $\mathcal{W}^0_\xi \cap \mathcal{S}_w$ be beautifully described? (regular?)
Note $\mathcal{W}^0_\xi \cap \mathcal{S}_w$ is regular for $g \leq 3$. At least we expect:

Expectation 5.1. $\mathcal{W}^0_{\xi(w)} \cap \mathcal{S}_w$ would be beautifully described.

Here $\xi(w)$ is the generic Newton polygon of $\mathcal{S}_w$. We have investigated the case $\xi(w) = \sigma$, i.e., $\mathcal{S}_w \subset \mathcal{W}_\sigma$:

Theorem 5.2 (H., to appear in J. Algebraic Geom.). *For any $w' \in \overline{W}'_c$ with $c \leq [g/2]$, there exists a finite surjective morphism*

$$G(\mathbb{Q})\backslash X(w') \times G(\mathbb{A}_f)/K \to \bigcup_{\mathfrak{r}(w)=w'} \mathcal{S}_w,$$

which is bijective on geometric points.

Here $X(w')$ is the (generalized) Deligne-Lusztig variety:

$$\{\mathrm{P} \in \mathrm{Sp}_{2c}/\mathrm{P}_0 \mid {}^h\mathrm{P} = \mathrm{P}_0, {}^h\mathrm{Fr}(\mathrm{P}) = {}^{w'}\mathrm{P}_0 \text{ for } \exists h \in \mathrm{Sp}_{2c}\},$$

and G is a certain quaternion unitary group over $\mathbb{Q}$.

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