# Lengths of chains of minimal rational curves on Fano manifolds

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Abstract

In this paper, we consider a natural question how many minimal rational curves are needed to join two general points on a Fano manifold of Picard number 1. In particular, we study the minimal length of such chains in the two cases where the dimension of $X$ is at most 5 and the coindex of $X$ is at most 3.

*2000 Mathematics Subject Classification*: 14E30, 14J45, 14N99.
*Keywords*: Fano manifold, chain of rational curves, minimal rational component, variety of minimal rational tangents.

## 1 Introduction

In this paper, we review some results announced at Kinosaki symposium without detailed proof.

For a Fano manifold, a *minimal rational component* $\mathscr{K}$ is defined to be a dominating irreducible component of the normalization of the parameter space of rational curves whose degree is minimal among such components and a *variety of minimal rational tangents* is the parameter space of the tangent directions of $\mathscr{K}$-curves at a general point. Nowadays these two objects often appear in the study of Fano manifolds [3, 10]. On the other hand, chains of rational curves play an important role in the study of Fano manifolds. For instance, Kollár-Miyaoka-Mori [14] and Nadel [16] independently showed the boundedness of the degree of Fano manifolds of Picard number 1 by using chains of rational curves. From these viewpoints, it is a natural question how many rational curves in the family $\mathscr{K}$ are needed to join two general points. We denote by $l_{\mathscr{K}}$ the minimal length of such chains of general $\mathscr{K}$-curves. In this direction, Hwang and Kebekus [4] developed an infinitesimal method to study the lengths of Fano manifolds via the varieties of minimal rational tangents. They also dealt with some examples when the varieties of minimal rational tangents and those secant varieties are simple, such as complete intersections, Hermitian symmetric spaces and homogeneous contact manifolds. Furthermore the following was obtained.

**Theorem 1.1** ([4, 6]). *Let $X$ be a prime Fano $n$-fold of Picard number $1$. If the Fano index $i_X$ satisfies $n+1 > i_X > \frac{2}{3}n$, then $l_{\mathscr{K}} = 2$.*

A Fano manifold is *prime* if the ample generator of the Picard group is very ample. Our original motivation of this paper is to announce a computational result of the

[1]The author is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. 30 June 2009.

lengths of Fano manifolds of coindex $\leq 3$. By the above theorem, it is sufficient to consider the cases where $n \leq 5$, $(n, i_X) = (6, 4)$ and $X$ is non-prime.

First we introduce the following:

**Theorem 1.2.** *Let $X$ be a Fano $n$-fold of Picard number 1, $\mathscr{K}$ a minimal rational component of $X$ and $p+2$ the anti-canonical degree of rational curves in $\mathscr{K}$. Then if $p = n-3 > 0$, we have $l_{\mathscr{K}} = 2$ and if $(n,p) = (5,1)$, we have $l_{\mathscr{K}} = 3$.*

By combining this theorem and well-known or easy arguments (see Section 3), we obtain the following table. In particular, when $n \leq 5$, $l_{\mathscr{K}}$ depends only on $(n,p)$.

Table A

| $n$ | $p$ | $l_{\mathscr{K}}$ | $n$ | $p$ | $l_{\mathscr{K}}$ | $n$ | $p$ | $l_{\mathscr{K}}$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 4 | 3 | 1 | 5 | 4 | 1 |
| 3 | 1 | 2 | 4 | 2 | 2 | 5 | 3 | 2 |
| 3 | 0 | 3 | 4 | 1 | 2 | 5 | 2 | 2 |
| | | | 4 | 0 | 4 | 5 | 1 | 3 |
| | | | | | | 5 | 0 | 5 |

On the other hand, the following shows $l_{\mathscr{K}}$ does not depend only on $(n,p)$ in general.

**Theorem 1.3.** *Let $X$ be a Fano manifold of Picard number 1 with coindex 3 and $\mathscr{K}$ a minimal rational component of $X$. Assume that $n := \dim X \geq 6$. Then $l_{\mathscr{K}} = 2$ except the case $X$ is a 6-dimensional Lagrangian Grassmannian $LG(3,6)$. In the case $X = LG(3,6)$, we have $l_{\mathscr{K}} = 3$.*

As a consequence, we obtain the following table ($n \geq 6$).

Table B

| $X$ | $i_X$ | $l_{\mathscr{K}}$ |
|---|---|---|
| $\mathbb{P}^n$ | $n+1$ | 1 |
| $Q^n$ | $n$ | 2 |
| del Pezzo mfd. of dim. $n$ | $n-1$ | 2 |
| Mukai mfd. of dim. $n \geq 7$ | $n-2$ | 2 |
| Mukai mfd. of dim. 6 | 4 | 2 or 3 |

The contents of this paper are organized as follows: In Sect. 2, we set up our notation and review some basic facts of deformation theory of rational curves. In Sect. 3, we explain why Table A and B follow from Theorem 1.2 and 1.3. In Sect. 4, we introduce a classification result of prime Fano $n$-folds satisfying $i_X = \frac{2}{3}n$ and $l_{\mathscr{K}} \neq 2$ as the next case of Theorem 1.1. Throughout this paper, we work over the complex number field.

## 2 Deformation theory of rational curves and varieties of minimal rational tangents

First we review some basic facts of deformation theory of rational curves and the definition of varieties of minimal rational tangents. For detail, we refer to [3, 12] and follow the conventions of them.

Throughout this paper, unless otherwise noted, we always assume that $X$ is a Fano manifold of $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathscr{O}_X(1)]$, where $\mathscr{O}_X(1)$ is the ample generator, and denote by $\mathrm{RatCurves}^n(X)$ the normalization of the space of integral rational curves on $X$. We also assume $n := \dim X \geq 3$. We denote by $i_X$ the *Fano index of* $X$ which is the integer satisfying $\omega_X \cong \mathscr{O}_X(-i_X)$, where $\omega_X$ is the canonical line bundle of $X$. We call $n+1-i_X$ the *coindex of* $X$.

As is well-known, a Fano manifold is uniruled. It is equivalent to the condition that there exists a free rational curve $f : \mathbb{P}^1 \to X$. Here we call a rational curve $f : \mathbb{P}^1 \to X$ *free* if $f^*T_X$ is semipositive. An irreducible component $\mathscr{K}$ of $\mathrm{RatCurves}^n(X)$ is called a *minimal rational component* if it contains a free rational curve of minimal anti-canonical degree. We denote by $\mathscr{K}_x$ the normalization of the subscheme of $\mathscr{K}$ parametrizing rational curves passing through $x$. Since each member of $\mathscr{K}$ is numerically equivalent, we can define the $\mathscr{O}_X(1)$-degree of $\mathscr{K}$ which is denoted by $d_{\mathscr{K}}$. We will use the symbol $p$ to denote $i_X d_{\mathscr{K}} - 2$. In this setting, the minimal rational component $\mathscr{K}$ satisfies the following fundamental properties.

**Proposition 2.1** (cf. [3]). (i) *For a general point $x \in X$, $\mathscr{K}_x$ is a disjoint union of smooth projective varieties of dimension $p$.*

(ii) *For a general member $[f]$ of $\mathscr{K}$, $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^p \oplus \mathscr{O}^{n-1-p}$ which is called a standard rational curve. In particular, $0 \leq p \leq n-1$.*

For a general point $x \in X$, we define the tangent map $\tau_x : \mathscr{K}_x \to \mathbb{P}(T_xX)$[2] by assigning the tangent vector at $x$ to each member of $\mathscr{K}_x$ which is smooth at $x$. We denote by $\mathscr{C}_x \subset \mathbb{P}(T_xX)$ the image of $\tau_x$, which is called the *variety of minimal rational tangents* at $x$.

**Theorem 2.2** ([5, 9]). *The tangent map $\tau_x : \mathscr{K}_x \to \mathscr{C}_x \subset \mathbb{P}(T_xX)$ is the normalization.*

**Theorem 2.3** ([2, 8]). *If $p = n-1$, namely $\mathscr{C}_x = \mathbb{P}(T_xX)$, then $X$ is isomorphic to $\mathbb{P}^n$.*

**Theorem 2.4** ([15]). *If $X$ is a Fano manifold of $n := \dim X \geq 3$, the following are equivalent.*

(i) *$X$ is isomorphic to a smooth quadric hypersurface $Q^n$.*

(ii) *The Picard number of $X$ is 1 and the minimal value of the anti-canonical degree of rational curves passing through a very general point $x_0 \in X$ is equal to $n$.*

**Corollary 2.5.** *If $p = n-2$, namely $\mathscr{C}_x \subset \mathbb{P}(T_xX)$ is a hypersurface, $X$ is isomorphic to $Q^n$.*

[2] For a vector space $V$, $\mathbb{P}(V)$ denotes the projective space of lines through the origin in $V$.

# 3 Table A and B

**Notation 3.1.** We denote by $(d_1) \cap \cdots \cap (d_k) \subset \mathbb{P}^n$ a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$, in particular, by $(d)^k$ if $d = d_1 = \cdots = d_k$. We denote by $G(k, n)$ a Grassmannian of $k$-planes in $\mathbb{C}^n$, by $LG(k, n)$ a Lagrangian Grassmannian which is the variety of isotropic $k$-planes for a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^n$, by $S_k$ the spinor variety which is an irreducible component of the Fano variety of $k$-planes in $Q^{2k}$. We denote a simple exceptional linear algebraic group of Dynkin type $G$ simply by $G$ and for a dominant integral weight $\omega$ of $G$, the minimal closed orbit of $G$ in $\mathbb{P}(V_\omega)$ by $G(\omega)$, where $V_\omega$ is the irreducible representation space of $G$ with highest weight $\omega$. For example, $E_7(\omega_1)$ is the minimal closed orbit of an algebraic group of type $E_7$ in $\mathbb{P}(V_{\omega_1})$, where $\omega_1$ is the first fundamental dominant weight in the standard notation of Bourbaki [1].

## 3.1 Table A

**Theorem 3.2.** (= Theorem 1.2) *If $p = n - 3 > 0$, we have $l_{\mathscr{K}} = 2$ and if $(n, p) = (5, 1)$, we have $l_{\mathscr{K}} = 3$.*

From Proposition 2.1, $p$ is at most $n - 1$ and at least 0. When $p = n - 1$, $X$ is isomorphic to $\mathbb{P}^n$ (Theorem 2.3). In particular, we have $l_{\mathscr{K}} = 1$. When $p = n - 2$, $X$ is isomorphic to $Q^n$ (Corollary 2.5). So if $p = n - 2$, we have $l_{\mathscr{K}} = 2$. Furthermore, if $p = 0$, we have $l_{\mathscr{K}} = n$. From these arguments, when $n$ is at most 5, it is enough to compute $l_{\mathscr{K}}$ in the cases where $(n, p) = (4, 1), (5, 2)$ and $(5, 1)$. However in these cases $l_{\mathscr{K}}$ can be computed from Theorem 3.2. Consequently, we obtain the following table:

Table A

| $n$ | $p$ | $l_{\mathscr{K}}$ | $n$ | $p$ | $l_{\mathscr{K}}$ | $n$ | $p$ | $l_{\mathscr{K}}$ | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 4 | 3 | 1 | 5 | 4 | 1 | $X = \mathbb{P}^n$ |
| 3 | 1 | 2 | 4 | 2 | 2 | 5 | 3 | 2 | $X = Q^n$ |
| 3 | 0 | 3 | 4 | 1 | 2 | 5 | 2 | 2 | |
| | | | 4 | 0 | 4 | 5 | 1 | 3 | |
| | | | | | | 5 | 0 | 5 | $p = 0 \Rightarrow l_{\mathscr{K}} = n$ |

## 3.2 Table B

**Theorem 3.3.** (= Theorem 1.3) *For a Fano manifold $X$ of Picard number 1 with coindex 3, assume that $n := \dim X \geq 6$. Then $l_{\mathscr{K}} = 2$ except the case $X$ is a 6-dimensional Lagrangian Grassmannian $LG(3, 6)$. In the case $X = LG(3, 6)$, we have $l_{\mathscr{K}} = 3$.*

Let $X$ be a Fano $n$-fold of Picard number 1 with coindex $\leq 3$. Since we obtained the list of $l_{\mathscr{K}}$ in the case $n \leq 5$ (Table A), assume $n$ is at least 6. If the coindex is equal to 0 or 1, then $X$ is a projective space or a smooth quadric hypersurface (Kobayashi-Ochiai Theorem [11]).

**Lemma 3.4.** *If the coindex is* 2*, we have* $p = n - 3$.

*Proof.* By our assumption, we have $i_X = n-1$. For a minimal rational component $\mathscr{K}$ of $X$, an equality $p+2 = i_X d_{\mathscr{K}}$ holds. Furthermore we know $n+1 \geq p+2 = (n-1)d_{\mathscr{K}}$. This implies that $d_{\mathscr{K}} = 1$. Thus we have $p = n - 3$. □

When the coindex is 2, we have $l_{\mathscr{K}} = 2$ from Theorem 3.2 and the above Lemma 3.4. When the coindex is 3 and $n \geq 6$, we know the value of $l_{\mathscr{K}}$ from Theorem 1.3. As a consequence, we obtain the following table ($n \geq 6$):

Table B

| $X$ | $i_X$ | $l_{\mathscr{K}}$ |
|---|---|---|
| $\mathbb{P}^n$ | $n+1$ | 1 |
| $Q^n$ | $n$ | 2 |
| del Pezzo mfd. of dim. $n$ | $n-1$ | 2 |
| Mukai mfd. of dim. $n \geq 7$ | $n-2$ | 2 |
| Mukai mfd. of dim. 6 | 4 | 2 or 3 |

From Table A and B, we can compute the lengths of Fano manifolds with coindex 3. In fact, we have the next table:

Table C (coindex 3 case)

| $n$ | $l_{\mathscr{K}}$ |
|---|---|
| $n \geq 7$ | 2 |
| 6 | 2 or 3 |
| 5 | 3 |
| 4 | 4 |
| 3 | 3 |

# 4 Boundary case

Before stating a classification result of prime Fano $n$-folds satisfying $i_X = \frac{2}{3}n$ and $l_{\mathscr{K}} \neq 2$, recall definitions and set up our notation.

**Definition 4.1** (cf. [7, 6])**.** For a projective manifold $X \subset \mathbb{P}^N$, we call $X$ *conic-connected* if there exists an irreducible conic passing through two general points on $X$.

**Example 4.2.** Let $X$ be a Grassmaniann $G(2,6) \subset \mathbb{P}^{14}$ or its linear section of dimension $n \geq 6$. Then $X$ is a Fano manifold of coindex 3 with the genus $g = 8$. For two distinct points $x, y \in G(2,6)$, they correspond to 2-dimensional vector subspaces $L_x, L_y \subset \mathbb{C}^6$. Then there exists a 4-dimensional vector subspace $V \subset \mathbb{C}^6$ which contains the join $< L_x, L_y >$. This implies that $x, y$ is contained in a 4-dimensional quadric $Q^4 \cong G(2,4) \subset G(2,6)$. So $X$ is conic-connected and $l_{\mathscr{K}} = 2$.

**Definition 4.3.** For a projective variety $X \subset \mathbb{P}^N$, we define the *secant variety of* $X$ by the closure of the union of lines passing through distinct two points on $X$ and denote by $S^1X$.

**Remark 4.4.** In general, it is easy to see the dimension of the secant variety $S^1X$ is at most $2n+1$. The expected dimension of the secant variety $S^1X$ is $2n+1$. When the dimension of $S^1X$ is less than $2n+1$, we say the secant variety $S^1X$ *defective*.

**Definition 4.5.** Let $X \subset \mathbb{P}^N$ be a non-degenerate smooth projective variety of dimension $n$. $X$ is a *Severi variety* if it satisfies that $3n = 2(N-2)$ and $S^1X \neq \mathbb{P}^N$.

**Theorem 4.6** ([17])**.** *Each Severi variety is projectively equivalent to one of the following:*

(i) *The Veronese surface* $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

(ii) *The Segre variety* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$.

(iii) *The Grassmann variety* $G(\mathbb{P}^1, \mathbb{P}^5) \subset \mathbb{P}^{14}$.

(iv) *The* $E_6$*-variety* $E_6(\omega_1) \subset \mathbb{P}^{26}$.

*In particular, Severi varieties are homogeneous.*

As an extremal case of Theorem 1.1, we have the following:

**Theorem 4.7.** *Let* $X$ *be a prime Fano* $n$*-fold with* $i_X = \frac{2}{3}n$*. Then* $l_{\mathscr{K}} = 2$ *except the following cases:*

(i) $(3) \subset \mathbb{P}^4$ *a hypersurface of degree* 3.

(ii) $(2) \cap (2) \subset \mathbb{P}^5$ *a complete intersection of two hyperquadrics.*

(iii) $G(2,5) \cap (1)^3 \subset \mathbb{P}^6$ *a* 3*-dimensional linear section of* $G(2,5)$.

(iv) $LG(3,6)$ *a Lagrangian Grassmannian.*

(v) $G(3,6)$ *a Grassmannian.*

(vi) $S_5$ *a spinor variety.*

(vii) $E_7(\omega_7)$ *a rational homogeneous manifold of type* $E_7$.

*In the cases* (i) – (vii) *we have* $l_{\mathscr{K}} = 3$.

**Corollary 4.8.** *Let* $X$ *be a prime Fano* $n$*-fold of Picard number* 1 *with* $i_X = \frac{2}{3}n$ *and* $\mathscr{K}$ *a minimal rational component of* $X$*. Assume that* $n \geq 6$*. Then the following are equivalent.*

(i) $l_{\mathscr{K}} \neq 2$.

(ii) $l_{\mathscr{K}} = 3$.

(iii) $X \subset \mathbb{P}(H^0(X, \mathscr{O}_X(1))^\vee)$ *is not conic-connected.*

(iv) *The dimension of the secant variety* $S^1X \subset \mathbb{P}(H^0(X, \mathscr{O}_X(1))^\vee)$ *is* $2n+1$.

(v) *The variety of minimal rational tangents at a general point* $\mathscr{C}_x \subset \mathbb{P}(T_xX)$ *is a Severi variety.*

(vi) $X \subset \mathbb{P}(H^0(X, \mathscr{O}_X(1))^\vee)$ *is projectively equivalent to one of the manifolds listed in Theorem 4.7* (iv) $-$ (vii).

This corollary and Theorem 1.1 implies that $i_X = \frac{2}{3}n$ is a boundary of conic-connectedness and defectiveness of the secant variety:

| Property | $i_X > \frac{2}{3}n$ | $i_X = \frac{2}{3}n$ | $i_X = \frac{2}{3}n$ |
|---|---|---|---|
| $l_{\mathscr{K}}$ | 2 | 2 | 3 |
| Conic-connectedness | Yes | Yes | No |
| Defectiveness of the secant variety | Yes | Yes | No |

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