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Gauss map of rank zero

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$X \subset \mathbb{P}^N$: a proj variety of dim n over $k = \bar{k}$ with $p = \mathrm{char}(k) > 0$.

Def (Gauss map). For an **embedding** $\iota : X \hookrightarrow \mathbb{P}^M$,

$$\gamma = \gamma_\iota : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^M) \quad (\textbf{Gauss map of } \iota)$$

is given by sending $x \in X$ to the embedded tangent T_xX in $\mathbb{P}^M$.
(We treat γ only for the case when $\iota(X)$ is non-linear in $\mathbb{P}^M$.)

Define **rk** γ (rank of the Gauss map) to be $\mathrm{rk}(d_x\gamma : t_xX \to t_{\gamma(x)}\mathbb{G}(n, \mathbb{P}^M))$ with gen $x \in X$.

- $p = 0 \Rightarrow \gamma$ is birational (thus, $\mathrm{rk}\,\gamma = n$) for any embedding ι if X is smooth.

In $p > 0$, birationality of γ is not guaranteed, and $\mathrm{rk}\,\gamma < n$ is possible.

We study the extreme case:

Def (GMRZ). We say that a proj variety X satisfies **(GMRZ)** if

$$\exists\, \iota : X \hookrightarrow \mathbb{P}^M \text{ s.t. } \mathrm{rk}\,\gamma_\iota = 0 \quad (\textbf{Gauss map of } \iota \text{ is of } \textbf{rank zero}).$$

Example (Fermat hypersurf). Let $X = (x_0^{p+1} + \cdots + x_N^{p+1} = 0) \subset \mathbb{P}^N$ $(p > 0)$.
Then $\gamma : \mathbb{P}^N \to \check{\mathbb{P}}^N : (x_0, \ldots, x_N) \mapsto (x_0^p, \ldots, x_N^p)$ (inseparable & $\mathrm{rk}\,\gamma = 0$).

Approach. Suppose that X has a **rational curve** param by $f : \mathbb{P}^1 \to X$.
If f is unramified, then the normal bdle N_f is given by $(\ker(f^*\Omega^1_X \to \Omega^1_{\mathbb{P}^1}))^\vee$.
If X satisfies (GMRZ), then N_f has the following significant property:

Theorem 1 (Splitting type of the normal bundle).
X: proj variety, $f : \mathbb{P}^1 \to X$: unramified. Assume that X is smooth along $f(\mathbb{P}^1)$, and $N_f^\vee \simeq \bigoplus_{i \geq -1} \mathcal{O}_{\mathbb{P}^1}(i)^{r_i}$ for some $r_i \in \mathbb{Z}_{\geqslant 0}$ $(i \geq -1)$. Then,

X satisfies (GMRZ) $\Rightarrow r_{i-1}r_i = 0$ for any $i \geq 0$.
Moreover, $r_i > 0$ for some $i \geq 0 \Rightarrow p = 2$ or $p | i + 1$.

Theorem 2. (1) $\prod_{1 \leq i \leq r} \mathbb{P}^{n_i}$ satisfies (GMRZ) $\Leftrightarrow p = 2$ & $n_i = 1$ $(\forall i)$.
(2) $\mathbb{G}(l, \mathbb{P}^m)$ satisfies (GMRZ) $\Leftrightarrow l = 0$ or $l = m - 1$.
(3) A smooth **quad** hypersurf in $\mathbb{P}^N$ $(N \geq 3)$ satisfies (GMRZ) $\Leftrightarrow p = 2$ & $N = 3$.
(4) A smooth **cubic** hypersurf in $\mathbb{P}^N$ $(N \geq 3)$ satisfies (GMRZ) $\Rightarrow p = 2$.

Theorem 3 (Cubic hypersurf). A smooth **cubic** hypersurf $X \subset \mathbb{P}^N$ with $N \geq 4$ satisfies **(GMRZ)**
$\Leftrightarrow p = 2$ and X is proj equivalent to a **Fermat** hypersurf.

Theorem 4. $X \subset \mathbb{P}^N$: a smooth **cubic** hypersurf with $N \geq 3$ in $p = 2$.
Denote by $\gamma_0 : X \to \check{\mathbb{P}}^N$ the Gauss map of the **original** embedding $X \subset \mathbb{P}^N$.
Then, $\mathrm{rk}\,\gamma_0 = 0 \Leftrightarrow X$ is proj equivalent to a **Fermat** hypersurf.

Rem. In order to prove Theorem 3, it is sufficient to show that "$\exists\, \iota : X \hookrightarrow \mathbb{P}^M$ s.t. $\mathrm{rk}\,\gamma_\iota = 0 \Rightarrow \mathrm{rk}\,\gamma_0 = 0$" due to Theorem 4. The case $N \geqslant 5$ was shown in [1], and the case $N = 4$ was recently shown in [2].

Theorem 5 (General hypersurf).
A **general** hypersurf in $\mathbb{P}^N$ of deg d with $3 \leqslant d \leqslant 2N - 3$ satisfies **(GMRZ)** $\Rightarrow p = 2$ and $d = 2N - 3$.

Rem. We have $N^\vee_{L/X} \simeq \mathcal{O}^{2N-3-d} \oplus \mathcal{O}(1)^{d-N+1}$ or $\mathcal{O}(-1)^{N-1-d} \oplus \mathcal{O}^{d-1}$ for a general **line** $L \subset X$ ([3, V, (4.4.2)]). Thus Thm 1 implies $d = 2N - 3$ or $d = N - 1$. By considering **conics** on X, we can show that $d = 2N - 3$.

Application (Absence of minimal free rational curves). $f : \mathbb{P}^1 \to X$ is said to be **free** if f^*T_X is generated by its global sections, and a free f **minimal** if $f^*T_X \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{d-2} \oplus \mathcal{O}^{n-d+1}$ with $d = \deg(-f^*K_X)$.

Theorem A [3, IV.2.10]. X: a smooth proj variety in $p = 0$. Then,
$\exists$ a **free** rational curve on $X \Rightarrow \exists$ a **minimal free** rational curve on X.

On the other hand, by using Theorem 1, we have:

Theorem 6. Assume $p > 0$, and let $X \subset \mathbb{P}^N$ be a **Fermat** hypersurf of deg $ep + 1$ $(e \in \mathbb{N})$. Then, (1) $N \geqslant 2ep + 1 \Rightarrow X$ has a **free** rat curve.
(2) $N > e(p + 1) \Rightarrow X$ has **no minimal free** rat curve.
Thus a Fermat hypersurf in $\mathbb{P}^N$ of deg $ep + 1$ with $N \geqslant 2ep + 1$ gives **a counter-example** for Theorem A in **each** $p > 0$.

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[2] K. Furukawa, preprint. [3] J. Kollár, Rational curves on algebraic varieties, Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, 3. Folge, Bd. **32**, Springer, Berlin, 1996.