代数幾何学シンポジウム記録
2010年度　pp.121–121

P-ADICALLY F-PURE TYPE SINGULARITY (AND QUASI P-ADICALLY F-PURE SINGULARITY)

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Introduction

Hara-Watanabe defined F-pure singularity of a pair (R, I^t), where R is a ring of characteristic $p > 0$, I an ideal of R and t a positive rational number. It seemed a positive characteristic analogue of log canonical singularity of a pair. In fact, for a ring R_0 of characteristic zero, an ideal I_0 and $t \in \mathbb{Q}_{>0}$, if a pair (R_0, I_0^t) is F-pure after reduction to p for infinitely many prime p, then the pair (R_0, I_0^t) should be log canonical! However, $(R, I^{\mathrm{fpt}(I)})$ is not necessary F-pure, although $(R_0, I_0^{\mathrm{lct}(I_0)})$ is always log canonical. The author wants to generalize F-pure singularity to satisfy that $(R, I^{\mathrm{fpt}(I)})$ is always F-pure.

$$F : R \to F_*R : \text{ Frobenius map}$$

Def 1 (Hara-Watanabe) (R, I^t) is F-pure if $\forall e \gg 0$, $\exists f \in I^{\lfloor t(p^e-1)\rfloor}$ such that

$$fF^e : R \to F_*^e R (x \mapsto fx^{p^e})$$

splits as an R-hom.

Def 2 (Takagi-Watanabe) F-pure thresholds are def. as

$$\mathrm{fpt}(I) := \sup\{t \in \mathbb{Q} | (R, I^t) \text{ is F-pure}\}$$
$$= \inf\{t \in \mathbb{Q} | (R, I^t) \text{ is not F-pure.}\}$$

We can compute F-purity and fpt(I) by the Following Fedder's type criterion: Suppose that $(R, \mathfrak{m})$ is regular local. For $f \in R$, $e \in \mathbb{N}$,

$$fF^e : R \to F_*^e R$$

splits iff

$$f \in \mathfrak{m}^{[p^e]} := (g^{p^e} | g \in \mathfrak{m}).$$

Ex 1 (F-pure thresholds)

$$I := (x^2 + y^3) \subseteq R := k[x, y]$$

p	fpt(I)	
2	1/2	
3	2/3	
$p \equiv 5 (\mathrm{mod} 6)$	$(5p-1)/6p$	
$p \equiv 1 (\mathrm{mod} 6)$	5/6	$= \mathrm{lct}(x^2 + y^3)$

For $I_0 \subseteq R_0$:an ideal of f.g. $\mathbb{C}$-algebra R_0, say $I_p \subseteq R_p$ is its reduction to char. p. (R_0, I_0^t) is F-pure type if $\exists$ infinitely many prime numbers p such that (R_p, I_p^t) is F-pure.

Thm 1 (Hara-Watanabe)
Assume that R_0: $\mathbb{Q}$-Gorenstein, normal over $\mathbb{C}$.

$$(R_0, I_0^t) : \text{ F-pure type} \Rightarrow \text{ log canonical.}$$

Hara proved $(R, (f)^{\mathrm{fpt}(f)})$:F-pure, (f): principal.

Ex 2 ($(R, I^{\mathrm{fpt}(I)})$: not noc. F-pure)

$$I := (x^2, y^2, z^2) \subseteq R := k[x, y, z]$$

Then $\forall p > 0$, fpt$(I) = \frac{3}{2}(= \mathrm{lct}(I))$

$p = 2$	$(R, I^{\mathrm{fpt}(I)})$: NOT F-pure
p: otherwise	$(R, I^{\mathrm{fpt}(I)})$: F-pure

$\nu(t) := -\min\{0, \nu_p(t)\}$, $\nu_p(-)$ is p-adic valuation

Def 3 (H) (R, I^t) is p-adically F-pure if $\exists$ infinitely many e with $p^{\nu(t)}t(p^e - 1) \in \mathbb{Z}$, $\exists f \in I^{p^{\nu(t)}t(p^e-1)}$ such that

$$fF^{e+\nu(t)} : R \to F_*^{e+\nu(t)} R (x \mapsto fx^{p^{e+\nu(t)}})$$

splits as an R-hom.

Rem 1 We don't need round down $\lfloor \ \rfloor$!
(We don't kill information of an exponent t.)

Prop 1 (H)

$$\mathrm{fpt}(I) = \inf\{t \in \mathbb{Q} | (R, I^t) \text{ is not p-adic. F-pure.}\}$$

Prop 2 (H)

$$\text{F-pure} \Rightarrow \text{p-adic. F-pure.}$$

Ex 3 I, R : same as in Ex2. Then $\forall p > 0$,

$$(R, I^{\mathrm{fpt}(I)}) \text{: p-adic. F-pure.}$$

Thm 2 (H) Suppose that $I = (f)$ is principal.

$$(R, (f)^{t'}) \text{ is p-adic. F-pure, } \forall t' \le t$$
$$\Rightarrow (R, (f)^t) \text{ is F-pure.}$$

p-adic. F-pure type can be defined like F-pure type.

Thm 3 (H) R_0: $\mathbb{Q}$-Gorenstein, normal over $\mathbb{C}$.

$$(R_0, I_0^t) : \text{ p-adic. F-pure type} \Rightarrow \text{log canonical.}$$

(Pf. of thm3.) Use Schwede's sharply F-purity;
(R, I^t) is sharply F-pure if $\exists$ infinitely many e, $\exists f \in I^{\lceil t(p^e-1)\rceil}$ such that

$$fF^e : R \to F_*^e R (x \mapsto fx^{p^e})$$

splits as an R-hom.
He proved sharply F-pure type implies log canonical. For fixed t, there is only finite prime numbers p such that $\nu_p(t) < 0$. If $\nu_p(t) \ge 0$, then sharp. F-pure = p-adic. F-pure. Therefore

$$(R_0, I_0^t) : \text{ p-adic. F-pure type} \Leftrightarrow \text{sharply F-pure type.}$$

(q.e.d.)

Ex 4 ($(R, I^{\mathrm{fpt}(I)})$: not nec. p-adic. F-pure)

$$I := (x^2, y^2) \subseteq R := k[x, y]$$

Then $\forall p > 0$, fpt$(I) = 1(= \mathrm{lct}(I))$.

$$(R, I^{\mathrm{fpt}(I)}) : \text{ p-adic. F-pure if } p \neq 2,$$
$$: \text{ quasi p-adic. F-pure } \forall p.$$

Def 4 (R, I^t) is quasi p-adically F-pure if $\exists \nu \ge 0$, $\exists$ infinitely many e with $p^\nu t(p^e - 1) \in \mathbb{Z}$, $\exists f \in I^{p^\nu t(p^e-1)}$ such that

$$fF^{e+\nu} : R \to F_*^{e+\nu} R (x \mapsto fx^{p^{e+\nu}})$$

splits as an R-hom.

Thm 4 (H) $R := k[x_1, \cdots, x_n]$, I its monomial ideal. Then

$$(R, I^{\mathrm{fpt}(I)}) \text{ is quasi p-adic. F-pure.}$$