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GREEN'S CONJECTURE FOR CURVES ON ARBITRARY $K3$ SURFACES

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Abstract: This is a report of my talk given at the Kinosaki Algebraic Geometry Symposium in October 2010, on the joint work with Marian Aprodu from IMAR Bucharest. The results described below will appear in the journal Compositio Mathematica in early 2011.

1. Introduction

Green's Conjecture on syzygies of canonical curves asserts that one can recognize existence of special linear series on an algebraic curve, by looking at the syzygies of its canonical embedding. Precisely, if C is a smooth algebraic curve of genus g, $K_{i,j}(C, K_C)$ denotes the (i,j)-th Koszul cohomology group of the canonical bundle K_C and Cliff(C) is the Clifford index of C, then M. Green [Gr84] predicted the vanishing statement

(1) $$K_{p,2}(C, K_C) = 0, \text{ for all } p < \text{Cliff}(C).$$

In recent years, Voisin [V02], [V05] achieved a major breakthrough by showing that Green's Conjecture holds for smooth curves C lying on $K3$ surfaces S with Pic$(S) = \mathbb{Z} \cdot C$. In particular, this establishes Green's Conjecture for general curves of every genus. Using Voisin's work, as well as a degenerate form of [HR98], it has been proved in [Ap05] that Green's Conjecture holds for any curve C of genus g of gonality gon$(C) = k \leq (g+2)/2$, which satisfies the *linear growth condition*

(2) $$\dim W^1_{k+n}(C) \leq n, \text{ for } 0 \leq n \leq g - 2k + 2.$$

Thus Green's Conjecture becomes a question in Brill-Noether theory. In particular, one can check that condition (2) holds for a general curve $[C] \in \mathcal{M}^1_{g,k}$ in any gonality stratum of $\mathcal{M}_g$, for all $2 \leq k \leq (g+2)/2$. Our main result is the following:

Theorem 1.1. *Green's Conjecture holds for every smooth curve C lying on an arbitrary $K3$ surface S.*

The proof of the statement does not cover, but it rather relies in an essential way on the most difficult case, that of curves of odd genus of maximal gonality. Precisely, when $g(C) = 2k-3$ and gon$(C) = k$, Theorem 1.1 as stated, is due to Voisin [V05] combined with results of Hirschowitz-Ramanan [HR98]. In the proof of Theorem 1.1, we distinguish two cases. When Cliff(C) is computed by a pencil (that is, Cliff$(C) =$ gon$(C) - 2$), we use a parameter count for spaces of Lazarsfeld-Mukai bundles [La86], [CP95], in order to find a smooth curve $C' \in |C|$ in the same linear system as C, such that C' verifies condition (2). Since Koszul cohomology satisfies the Lefschetz hyperplane principle, one has that $K_{p,2}(C, K_C) \cong K_{p,2}(C', K_{C'})$. This proves Green's Conjecture for C.

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When Cliff(C) is no longer computed by a pencil, it follows from [Kn09] that C is a *generalized ELMS example*, in the sense that there exist smooth curves $D, \Gamma \subset S$, with $\Gamma^2 = -2, \Gamma \cdot D = 1$ and $D^2 \geq 2$, such that $C \equiv 2D + \Gamma$ and Cliff$(C) =$ Cliff$(\mathcal{O}_C(D)) = \text{gon}(C) - 3$. This case requires a separate analysis since condition (2) is no longer satisfied, and we refer to Section 4 for details.

Theorem 1.1 follows by combining results obtained by using the powerful techniques developed in [V02], [V05], with facts about the effective cone of divisors of $\overline{\mathcal{M}}_g$. As pointed out in [Ap05], starting from a k-gonal smooth curve $[C] \in \mathcal{M}_g$ satisfying the Brill-Noether growth condition (2), by identifying pairs of general points $x_i, y_i \in C$ for $i = 1, \ldots, g+3-2k$ one creates a *stable* curve

$$\big[X := C/x_1 \sim y_1, \ldots, x_{g+3-2k} \sim y_{g+3-2k}\big] \in \overline{\mathcal{M}}_{2g+3-2k}$$

having maximal gonality $g+3-k$, that is, lying outside the closure of the Hurwitz divisor $\mathcal{M}^1_{2g+3-2k,g+3-k}$ consisting of curves with a pencil $\mathfrak{g}^1_{g+3-k}$. Since the class of the virtual failure locus of Green's Conjecture is a multiple of the Hurwitz divisor $\overline{\mathcal{M}}^1_{2g+3-2k,g+3-k}$ on $\overline{\mathcal{M}}_{2g+3-2k}$, see [HR98], Voisin's theorem can be extended to all irreducible stable curves of genus $2g+3-2k$ and having maximal gonality, in particular to X as well, and a posteriori to smooth curves of genus g sitting on $K3$ surfaces with arbitrary Picard lattice. On the other hand, showing that condition (2) is satisfied for a curve $[C] \in \mathcal{M}_g$, is a question of pure Brill-Noether nature.

Theorem 1.1 has strong consequences on Koszul cohomology of $K3$ surfaces. It is known that for any globally generated line bundle L on a $K3$ surface S, the Clifford index of any smooth irreducible curve is constant, equal to, say c, [GL87]. Applying Theorem 1.1, Green's hyperplane section theorem, the duality theorem and finally the Green-Lazarsfeld nonvanishing theorem [Gr84], we obtain a complete description of the distribution of zeros among the Koszul cohomology groups of S with values in L.

Theorem 1.2. *Suppose $L^2 = 2g-2 \geq 2$. The Koszul cohomology group $K_{p,q}(S, L)$ is nonzero if and only if one of the following cases occur:*

(1) $q = 0$ *and* $p = 0$, *or*
(2) $q = 1$, $1 \leq p \leq g-c-2$, *or*
(3) $q = 2$ *and* $c \leq p \leq g-1$, *or*
(4) $q = 3$ *and* $p = g-2$.

2. Brill-Noether loci and their dimensions

Throughout this section we fix a $K3$ surface S, a globally generated line bundle $L \in \text{Pic}(S)$ and denote by $|L|_s$ the locus of smooth connected curves in $|L|$. For integers $r, d \geq 1$, we consider the morphism $\pi_S : \mathcal{W}^r_d(|L|) \to |L|_s$ with fibre over a point $C \in |L|_s$ isomorphic to the Brill-Noether locus $W^r_d(C)$. The analysis of the Brill-Noether loci $W^r_d(C)$ for a *general curve* $C \in |L|$ in its linear system, is equivalent to the analysis of the restricted maps $\pi_S : \mathcal{W} \to |L|$ over irreducible components $\mathcal{W}$ of $\mathcal{W}^r_d(|L|)$ dominating the linear system.

The main ingredient used to study $\mathcal{W}^r_d(|L|)$ is the *Lazarsfeld-Mukai bundle* [La86] associated to a complete linear series. To any pair (C, A) consisting of a curve $C \in |L|_s$ and a base point free linear series $A \in W^r_d(C) \setminus W^{r+1}_d(C)$, one associates the *Lazarsfeld-Mukai bundle* $E_{C,A} := F^\vee_{C,A}$ on S, via an elementary transformation along $C \subset S$:

(3) $$0 \to F_{C,A} \to H^0(C,A) \otimes \mathcal{O}_S \stackrel{\text{ev}}{\to} A \to 0.$$

Dualizing the sequence (3), we obtain the short exact sequence

(4) $$0 \to H^0(C,A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to K_C \otimes A^\vee \to 0.$$

When $C \in |L|_s$ and $A \in W^r_d(C) \setminus W^{r+1}_d(C)$ is globally generated, we consider the Petri map

$$\mu_{0,A} : H^0(C,A) \otimes H^0(C, K_C \otimes A^\vee) \to H^0(C, K_C),$$

whose kernel can be described in terms of Lazarsfeld-Mukai bundles. Let M_A the vector bundle of rank r on C defined as the kernel of the evaluation map

(5) $$0 \to M_A \to H^0(C,A) \otimes \mathcal{O}_C \stackrel{\text{ev}}{\to} A \to 0.$$

Since $h^0(C, F_{C,A}) = h^1(C, F_{C,A}) = 0$, one writes that

$$H^0(C, E_{C,A} \otimes F_{C,A}) = H^0(C, F_{C,A} \otimes K_C \otimes A^\vee).$$

We shall use the following deformation-theoretic result [Pare95], which is a consequence of Sard's theorem applied to the projection $\pi_S : \mathcal{W}^r_d(|L|) \to |L|$.

Lemma 2.1. *Suppose $\mathcal{W} \subset \mathcal{W}^r_d(|L|)$ is a dominating component, and $(C,A) \in \mathcal{W}$ is a general element such that A is globally generated and $h^0(C,A) = r+1$. Then the coboundary map $H^0(C, M_A \otimes K_C \otimes A^\vee) \to H^1(C, \mathcal{O}_C)$ is zero.*

The above analysis can be summarized as follows:

Proposition 2.2. *If $\mathcal{W} \subset \mathcal{W}^r_d(|L|)$ is a dominating component, and $(C,A) \in \mathcal{W}$ is a general element such that A is globally generated and $h^0(C,A) = r+1$, then $\dim_A W^r_d(C) \leq \rho(g,r,d) + h^0(C, E_{C,A} \otimes F_{C,A}) - 1$. Moreover, equality holds if $\mathcal{W}$ is reduced at (C,A).*

In particular, if $E_{C,A}$ is a simple bundle, then $\mu_{0,A}$ is injective and $\mathcal{W}$ is reduced at (C,A) of dimension $\rho(g,r,d) + g$. Thus, the problem of estimating $\dim_A W^r_d(C)$, when $(C,A) \in \mathcal{W}$ is suitably general, can be reduced to the case when $E_{C,A}$ is *not* a simple bundle.

3. Varieties of pencils on $K3$ sections

If L is a line bundle on a smooth projective variety X and $L \in \text{Pic}(X)$ is a line bundle, we write $L \geq 0$ when $H^0(X,L) \neq 0$. If E is a vector bundle on X and $L \in \text{Pic}(X)$, we set $E(-L) := E \otimes L^\vee$.

As in the previous section, we fix a $K3$ surface S together with a globally generated line bundle L on S. We denote by k be the minimal gonality of smooth curves in the linear system $|L|$, and set $g := 1 + L^2/2$. Suppose that $\rho(g,1,k) \leq 0$ (this leaves out one single case, namely $g = 2k-3$, when $\rho(g,1,k) = 1$). Our aim is to prove the Koszul vanishing statement

$$K_{g-\text{Cliff}(C)-1,1}(C, K_C) = 0,$$

for any curve $C \in |L|_s$. By duality, this is equivalent to Green's Conjecture for C.

It was proved in [Ap05] that any smooth curve C that satisfies the linear growth condition (2), verifies both Green's and Green-Lazarsfeld Gonality Conjecture. By comments made in the previous section, a general curve $C \in |L|_s$ satisfies (2), if and only if for any $n = 0, \ldots, g-2k+2$, and any irreducible component $W \subset W^1_{k+n}(C)$ such that a general element $A \in W$ is globally generated, has

$h^0(C,A) = 2$, and the corresponding Lazarsfeld-Mukai bundle $E_{C,A}$ is not simple, the estimate $\dim W \le n$, holds.

Condition (2) for curves which are general in their linear system, can be verified either by applying Proposition 2.2, or by estimating directly the dimension of the corresponding irreducible components of the scheme $\mathcal{W}^1_{k+n}(|L|)$. In our analysis, we need the following description [DM89] of non-simple Lazarsfeld-Mukai bundles, see also [CP95] Lemma 2.1:

Lemma 3.1. *Let $E_{C,A}$ be a non-simple Lazarsfeld-Mukai bundle. Then there exist line bundles $M, N \in \mathrm{Pic}(S)$ such that $h^0(S,M),\ h^0(S,N) \ge 2$, N is globally generated, and there exists a zero-dimensional, locally complete intersection subscheme ξ of S such that $E_{C,A}$ is expressed as an extension*

$$0 \to M \to E_{C,A} \to N \otimes I_\xi \to 0. \tag{6}$$

Moreover, if $h^0(S, M \otimes N^\vee) = 0$, then $\xi = \emptyset$ and the extension splits.

We say that (6) is the *Donagi-Morrison* (DM) extension associated to $E_{C,A}$. The size of the space of endomorphisms of a non-simple Lazarsfeld-Mukai bundle can be explicitly computed from the corresponding DM extension:

Lemma 3.2. *Let E be a non-simple Lazarfeld-Mukai bundle on S with $\det(E) = L$, and M and N the corresponding line bundles from the DM extension. If E is indecomposable, then*

$$h^0(S, E \otimes E^\vee) = 1 + h^0(S, M \otimes N^\vee).$$

If $E = M \oplus N$, then $h^0(S, E \otimes E^\vee) = 2 + h^0(S, M \otimes N^\vee) + h^0(S, N \otimes M^\vee)$.

In order to parameterize all pairs (C, A) with non-simple Lazarsfeld-Mukai bundles, we need a global construction. We fix a non-trivial globally generated line bundle N on S such that $H^0(S, L(-2N)) \neq 0$, and an integer $\ell \ge 0$. We set $M := L(-N)$ and $g := 1 + L^2/2$. Define $\widetilde{\mathcal{P}}_{N,\ell}$ to be the family of *vector bundles* of rank 2 on S given by non-trivial extensions

$$0 \to M \to E \to N \otimes I_\xi \to 0, \tag{7}$$

where ξ is a zero-dimensional lci subscheme of S of length ℓ, and set

$$\mathcal{P}_{N,\ell} := \{[E] \in \widetilde{\mathcal{P}}_{N,\ell} :\ h^1(S,E) = h^2(S,E) = 0\}.$$

Equivalently (by Riemann-Roch), $[E] \in \mathcal{P}_{N,\ell}$ if and only if $h^0(S,E) = g - c_2(E) + 3$ and $h^1(S,E) = 0$.

Assuming that $\mathcal{P}_{N,\ell} \neq \emptyset$, we consider the Grassmann bundle $\mathcal{G}_{N,\ell}$ over $\mathcal{P}_{N,\ell}$ classifying pairs (E, Λ) with $[E] \in \mathcal{P}_{N,\ell}$ and $\Lambda \in \mathrm{G}(2, H^0(S,E))$. If $d := c_2(E)$ we define the rational map $h_{N,\ell} : \mathcal{G}_{N,\ell} \dashrightarrow \mathcal{W}^1_d(|L|)$, by setting $h_{N,\ell}(E,\Lambda) := (C_\Lambda, A_\Lambda)$, where $A_\Lambda \in \mathrm{Pic}^d(C_\Lambda)$ is such that the following exact sequence on C_Λ holds:

$$0 \to \Lambda \otimes \mathcal{O}_S \overset{\mathrm{ev}_\Lambda}{\to} E \to K_{C_\Lambda} \otimes A_\Lambda^\vee \to 0.$$

Lemma 3.3. *If $\mathcal{P}_{N,\ell} \neq \emptyset$, then* $\dim \mathcal{G}_{N,\ell} = g + \ell + h^0(S, M \otimes N^\vee)$.

Lemma 3.4. *Suppose that a smooth curve $C \in |L|$ has Clifford dimension one and A is a globally generated line bundle on C with $h^0(C,A) = 2$ and $[E_{C,A}] \in \mathcal{P}_{N,\ell}$. Then $M \cdot N \ge \mathrm{gon}(C)$.*

Proof. One knows that $M|_C$ contributes to the Cliff(C). From the exact sequence $0 \to N^\vee \to M \to M|_C \to 0$ and from the observation that $h^1(S,N)=0$, we obtain by direct computation that

$$\text{Cliff}(M|_C) = M\cdot N + M^2 - 2h^0(S,M) + 2 = M \cdot N - 2 - 2h^1(S,M) \geq k-2,$$

that is, $M \cdot N \geq k + 2h^1(S,M) \geq k$. □

This takes care of indecomposable non-simple Lazarsfeld-Mukai bundles, and computes the dimensions of the corresponding parameter spaces. We can summarize our results as follows:

Theorem 3.5. *Let S be a $K3$ surface and L a globally generated line bundle on S, such that general curves in $|L|$ are of Clifford dimension one. Suppose that $\rho(g,1,k) \leq 0$, where $L^2 = 2g-2$ and k is the (constant) gonality of all curves in $|L|_s$. Then a general curve $C \in |L|$ satisfies the linear growth condition (2), thus Green's Conjecture is verified for* any *smooth curve in $|L|$.*

In the case $\rho(g,1,k)=1$, Green's Conjecture is also verified for smooth curves in $|L|$, cf. [V05], [HR98]. To sum up, Green's Conjecture is verified for any curve of Clifford dimension one on a $K3$ surfaces.

4. Curves of higher Clifford dimension

We begin with a preparatory result:

Theorem 4.1. *Let C be a smooth curve of Clifford dimension one and $x, y \in C$ be distinct points, and denote*

$$\mathcal{Z}_n := \{A \in W^1_{k+n}(C) : \; h^0(C, A(-x-y)) \geq 1\}.$$

Suppose that $\dim \mathcal{Z}_n \leq n-1$, *for all* $0 \leq n \leq g-2k+2$. *Then the bundle $K_C(x+y)$ verifies the Gonality Conjecture.*

The condition in the statement of Theorem 4.1 means that passing through the points x and y is a non-trivial condition on any irreducible component of maximal allowed dimension n of the Brill-Noether locus $W^1_{k+n}(C)$, for all $0 \leq n \leq g-2k+2$.

We now turn to the analysis of the Koszul cohomology of curves of higher Clifford dimension on a $K3$ surface S. Since plane curves are known to verify Green's Conjecture, the significant cases occur when the Clifford dimension is at least 3. Note that, unlike the Clifford index, the Clifford dimension is *not* semi-continous. An example was given by Donagi-Morrisson [DM89]: If $\epsilon : S \to \mathbf{P}^2$ is a double sextic and $L = \epsilon^*(\mathcal{O}_{\mathbb{P}^2}(3))$, then the general element in $|L|$ is isomorphic to a smooth plane sextic, hence it has Clifford dimension 2, while special points correspond to bielliptic curves and are of Clifford dimension 1.

It was proved in [CP95] that, except for the Donagi-Morrisson example, if a globally generated linear system $|L|$ on S contains smooth curves of Clifford dimension at least 2, then $L = \mathcal{O}_S(2D+\Gamma)$, where $D, \Gamma \subset S$ are smooth curves, $D^2 \geq 2$ (hence $h^0(S, \mathcal{O}_S(D)) \geq 2$), $\Gamma^2 = -2$ and $D \cdot \Gamma = 1$. If the genus of D is $r \geq 3$, then the genus of a smooth curve $C \in |L|$ equals $4r-2 \geq 10$, and gon$(C) = 2r$, while Cliff$(C) = 2r-3$; the Clifford dimension of C is r. From now on, we assume that we are in this situation.

Green's hyperplane section theorem implies that the Koszul cohomology is constant in a linear system. We degenerate a smooth curve $C \in |2D+\Gamma|$ to a reducible curve $X + \Gamma$ with $X \in |2D|$. In order to be able to carry out this plan, we first analyze the geometry of the curves in $|2D|$. Notably, we shall prove:

Theorem 4.2. *The hypothesis of Theorem 4.1 are verified for a general curve $X \in |2D|$ and the two points of intersection $X \cdot \Gamma$.*

The proof of Theorem 4.2 proceeds in several steps. The first result describes the fundamental invariants of a quadratic complete intersection section of S:

Lemma 4.3. *Any smooth curve $X \in |2D|$ has genus $4r-3$, gonality $2r-2$, and $\mathrm{Cliff}(X) = 2r-4$.*

It suffices therefore to analyze the structure of the loci $W^1_{2r-2+n}(X)$ where $n \leq 3 = g(X) - 2\mathrm{gon}(X) + 2$, and more precisely those components of dimension n.

Lemma 4.4. *We fix a general $X \in |2D|$, viewed as a half-canonical curve $X \xrightarrow{|D|} \boldsymbol{P}^r$.*

- *$W^1_{2r-2}(X)$ is finite and all minimal pencils $\mathfrak{g}^1_{2r-2}$ on X are given by the rulings of quadrics of rank 4 in $H^0(\boldsymbol{P}^r, \mathcal{I}_{X/\boldsymbol{P}^r}(2))$.*
- *X has no base point free pencils $\mathfrak{g}^1_{2r-1}$, that is, $W^1_{2r-1}(X) = X + W^1_{2r-2}(X)$.*
- *For $n = 2, 3$, if $A \in W^1_{2r-2+n}(X)$ is a base point free pencil, then the vector bundle $E_{X,A}$ is not simple.*

In all cases $n \leq 3$, if A belongs to an n-dimensional component of $W^1_{2r-2+n}(X)$, then the corresponding DM extension

$$0 \to M \to E_{X,A} \to N \otimes I_\xi \to 0$$

verifies $\mathrm{length}(\xi) = n$, *$M \cdot N = 2r-2$ and $M \cdot \Gamma = N \cdot \Gamma = 1$. When $n = 2, 3$, we can take $M = N = \mathcal{O}_X(D)$.*

The main result of this section is the following:

Theorem 4.5. *Smooth curves of Clifford dimension at least three on $K3$ surfaces satisfy Green's Conjecture.*

Proof. As in [ApP08, Section 4.1], for all $p \geq 1$, we have isomorphisms

$$K_{p,1}(X + \Gamma, \omega_{X+\Gamma}) \cong K_{p,1}(X, K_X(\Gamma)).$$

One finds that $K_{2r,1}(X + \Gamma, \omega_{X+\Gamma}) = 0$, implying the vanishing of $K_{2r,1}(S, L)$, via Green's hyperplane section theorem. Using the hyperplane section theorem again, we obtain $K_{2r,1}(C, K_C) = 0$, for any smooth curve $C \in |L|$, that is, the vanishing predicted by Green's Conjecture for C. □

Theorems 3.5 and 4.5, and the main results of [V05] and [HR98], altogether complete the proof of Theorem 1.1.

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