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(UNI)RATIONALITY OF THE MODULI SPACES OF 2-ELEMENTARY $K3$ SURFACES

SHOUHEI MA

ABSTRACT. We review our study on the birational type of the moduli spaces of $K3$ surfaces with non-symplectic involution. The main results are that all the moduli spaces are unirational, and that many of them are in fact rational.

1. 2-ELEMENTARY $K3$ SURFACE

Let us begin with basic definitions.

Definition 1.1. Let X be a complex $K3$ surface. An involution ι on X is *non-symplectic* if ι acts by -1 on $H^0(K_X)$. For such an ι we call the pair (X, ι) a *2-elementary* $K3$ *surface*.

For a 2-elementary $K3$ surface (X, ι) the underlying surface X is algebraic, and the fixed locus $X^\iota = \{x \in X, \iota(x) = x\}$ is a disjoint union of smooth curves. The invariant lattice $L_+ = \{l \in H^2(X, \mathbb{Z}),\ \iota^* l = l\}$, equipped with the intersection form, is an even lattice of signature $(1, \mathrm{rk} L_+ - 1)$. When $X^\iota = \emptyset$, the quotient surface $Y = X/\langle\iota\rangle$ is an Enriques surface. From this point of view 2-elementary $K3$ surfaces may be regarded as generalizations of Enriques surfaces. When $X^\iota \neq \emptyset$, Y is a smooth rational surface, and the quotient morphism $X \to Y$ is a double cover branched along a smooth $-2K_Y$-curve on Y. The double covers of $Y = \mathbb{P}^2$ branched along smooth sextics are the most basic 2-elementary $K3$ surfaces.

We shall define invariants of (X, ι) by using the lattice L_+. The discriminant group $D_{L_+} = L_+^\vee / L_+$ of L_+ is a 2-elementary Abelian group, namely $D_{L_+} \simeq (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \geq 0$. The quadratic form on the dual lattice $L_+^\vee$ induces the discriminant form $q : D_{L_+} \to \mathbb{Q}/2\mathbb{Z}, q(x + L_+) = (x, x) + 2\mathbb{Z}$. When $q(D_{L_+}) \subset \mathbb{Z}/2\mathbb{Z}$, we say that q has parity $\delta(q) = 0$, and in other cases we say that q has parity $\delta(q) = 1$.

Definition 1.2. The *main invariant* of a 2-elementary $K3$ surface (X, ι) is the triplet (r, a, δ) where r is the rank of L_+, a is the length of D_{L_+}, and δ is the parity of q.

The main invariant is related to the topology of the fixed curve X^ι.

Proposition 1.3 (Nikulin [10]). *Let (r, a, δ) be the main invariant of a 2-elementary $K3$ surface (X, ι). If $(r, a, \delta) = (10, 10, 0)$, then $X^\iota = \emptyset$. If $(r, a, \delta) = (10, 8, 0)$, then X^ι is a union of two elliptic curves. For other main invariants, X^ι is decomposed as $X^\iota = C^g \sqcup E_1 \sqcup \cdots \sqcup E_k$ where C^g is a genus g curve and $E_1, \cdots, E_k$ are (-2)-curves with*

$$g = 11 - \frac{r + a}{2}, \qquad k = \frac{r - a}{2}. \tag{1.1}$$

One has $\delta = 0$ if and only if the class of X^ι is divisible by 2 in NS_X.

2. Classification and moduli spaces

Nikulin classified 2-elementary $K3$ surfaces in terms of the main invariants.

Theorem 2.1 (Nikulin [10]). *The deformation type of a 2-elementary $K3$ surface (X, ι) is determined by the main invariant (r, a, δ). All possible main invariants of 2-elementary $K3$ surfaces are seventy-five in number, and are shown on the Figure 1.*

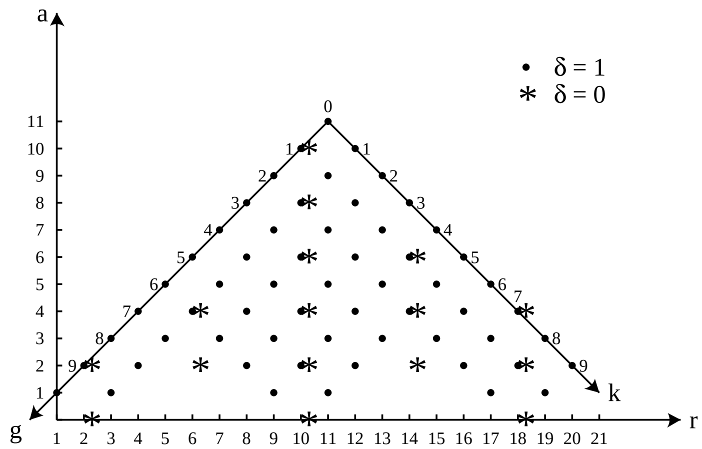

Figure 1. Geography of main invariants (r, a, δ)

A moduli space of 2-elementary $K3$ surfaces of fixed main invariant was constructed by Yoshikawa. Let (X, ι) be an arbitrary 2-elementary $K3$ surface of type (r, a, δ) and $L_- = L_+^\perp \cap H^2(X, \mathbb{Z})$ be the lattice of ι-anti-invariant cycles, which has the signature $(2, 20 - r)$. To such a lattice L_- is associated a Hermitian symmetric domain Ω_{L_-}. We let $\mathcal{F}(O(L_-)) = O(L_-)\backslash\Omega_{L_-}$ be the modular variety of type IV associated to $O(L_-)$. The orthogonal complements of the (-2)-vectors of L_- in Ω_{L_-} define a Heegner divisor $H \subset \mathcal{F}(O(L_-))$. Let $\mathcal{M}_{(r,a,\delta)}$ be the complement

$$\mathcal{M}_{(r,a,\delta)} = \mathcal{F}(O(L_-)) - H, \tag{2.1}$$

which is a normal, irreducible, and quasi-projective variety of dimension $20 - r$.

Theorem 2.2 ([12], [13])**.** *The variety $\mathcal{M}_{(r,a,\delta)}$ is a moduli space of 2-elementary $K3$ surfaces of type (r, a, δ).*

The object of this talk is the birational type of the moduli varieties $\mathcal{M}_{(r,a,\delta)}$. There are several known results. The 2-elementary $K3$ surfaces constructed from smooth plane sextics belong to $\mathcal{M}_{(1,1,1)}$. Hence $\mathcal{M}_{(1,1,1)}$ is birational to the orbit space $|\mathcal{O}_{\mathbb{P}^2}(6)|/\mathrm{PGL}_3$, which is clearly unirational. Kondō [6] proved the rationality of $\mathcal{M}_{(10,2,0)}$ and $\mathcal{M}_{(10,10,0)}$, the latter being isomorphic to the moduli of Enriques surfaces. The rationality of $\mathcal{M}_{(5,5,1)}$ was practically established in the work of Shepherd-Barron [11]. By the work of Matsumoto-Sasaki-Yoshida [9] on six lines on $\mathbb{P}^2$, $\mathcal{M}_{(16,6,1)}$ is known to be unirational. The work of Koike-Shiga-Takayama-Tsutsui [5] related to [9] shows that $\mathcal{M}_{(14,8,1)}$ is unirational. On the other hand, Yoshikawa [14] found that $\mathcal{M}_{(r,a,\delta)}$ has Kodaira dimension $-\infty$ if either $13 \leq r \leq 17$ or $r + a = 22$, $r \leq 17$, by using modular forms on $\mathcal{M}_{(r,a,\delta)}$.

3. Main results

Our main results are the following.

Theorem 3.1 ([7])**.** *For every main invariant (r, a, δ) the moduli space $\mathcal{M}_{(r,a,\delta)}$ is unirational.*

Theorem 3.2 ([8])**.** *The moduli space $\mathcal{M}_{(r,a,\delta)}$ is rational if (r, a, δ) is in the following range.*

(1) $2 \leq g \leq 9$, $\delta = 1$, $(g, k) \neq (2, 1)$.
(2) $g \leq 1$, $\delta = 1$, $g + k \geq 5$.
(3) $\delta = 0$, $(g, k) \neq (9, 0)$.

There are sixty-three main invariants in this range.

I do not know whether the remaining twelve moduli spaces beyond Theorem 3.2 are rational or irrational. They are possibly all rational, but I have no convincing evidence.

In the rest of this talk, I explain the idea of the proof of these theorems. I found a relatively short and systematic proof for Theorem 3.1, while the proof of Theorem 3.2 is rather ad hoc and long. So I give the proof for these two theorems separately. Of course one can prove Theorem 3.1 by just supplementing Theorem 3.2, but if we do so, the whole proof of unirationality would be very lengthy. I here prefer the more systematic and self-contained proof.

4. Proof of unirationality

Roughly speaking, I construct isogenies between certain finite Galois covers of the moduli spaces to reduce the unirationality problem to those covers of fewer moduli spaces. Here is a more precise strategy. Let L_- be the lattice of signature $(2, 20 - r)$ used in the construction of $\mathcal{M}_{(r,a,\delta)}$ and $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ be the modular variety associated to the group $O(L_-)_0$ of isometries of L_- which act trivially on the discriminant group of L_-. Since $O(L_-)_0$ is a finite-index subgroup of $O(L_-)$, the variety $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ is a finite Galois cover of the moduli space $\mathcal{M}_{(r,a,\delta)}$. We proceed as follows.

(1) Construct a finite surjective morphism $\widetilde{\mathcal{M}}_{(r,a,\delta)} \to \widetilde{\mathcal{M}}_{(r,a',\delta')}$ when either $a' < a$, $\delta = 1$, or $a' < a$, $\delta = \delta'$.
(2) For each fixed $1 \leq r \leq 19$, choose a large a and find a moduli interpretation of (an open set of) $\widetilde{\mathcal{M}}_{(r,a,\delta)}$.
(3) Prove the unirationality of $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ by using the moduli interpretation. By the step (1) follows the unirationality of $\widetilde{\mathcal{M}}_{(r,a',\delta')}$ for $a' < a$.
(4) The remaining moduli spaces $\mathcal{M}_{(r,a'',\delta'')}$, $a'' > a$, are also proved to be unirational in some way. This concludes the proof.

The step (1) is the key step. It reduces the problem to the covers $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ with large a. The isogeny is constructed through an embedding of the arithmetic groups, and it admits a geometric interpretation in terms of twisted Fourier-Mukai partner of $K3$ surfaces (see [7] for the detail). The cover $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ parametrizes 2-elementary $K3$ surfaces with some additional structure. More specifically, $\widetilde{\mathcal{M}}_{(r,a,\delta)}$ is the so-called "moduli of lattice-polarized $K3$ surfaces" (see [3]). However, to cope with the unirationality problem for $\widetilde{\mathcal{M}}_{(r,a,\delta)}$, we leave from its interpretation in terms of $K3$ and lattice, and seek for another more geometric interpretation. For example, $\widetilde{\mathcal{M}}_{(r,r,1)}$ is shown to be birational to a natural $\mathfrak{S}_{r-1}$-cover of the Severi variety of irrducible $(r - 1)$-nodal plane sextics. On the other side, $\widetilde{\mathcal{M}}_{(r,22-r,\delta)}$ with $r \geq 14$ turns out to be birational to a configuration space of point set in $\mathbb{P}^2$, which I shall explain in the next section.

5. Period maps of orthogonal type for $5 \leq d \leq 8$ point sets in $\mathbb{P}^2$

When I sought for moduli interpretation for $\widetilde{\mathcal{M}}_{(r,22-r,\delta)}$ with $r \geq 12$, I found as by-product period maps for $5 \leq d \leq 8$ point sets in $\mathbb{P}^2$ with values in modular varieties of type IV. Let $U_d \subset (\mathbb{P}^2)^d$ (resp. $V_d \subset (\mathbb{P}^2)^d$) be the variety of ordered d points of which no three are collinear (resp. only the first three are collinear). By using GIT, one sees that there exist geometric quotients U_d/G and V_d/G for the diagonal actions of $G = \mathrm{PGL}_3$. Let L_n be the lattice $\langle 2\rangle^2 \oplus \langle -2\rangle^n$. My result is stated as follows.

Theorem 5.1. *Let $5 \leq d \leq 8$. For each $1 \leq n \leq 8$ there exists an arithmetic group $\Gamma_n \subset O(L_n)$ such that one has birational period maps*

$$U_d/G \dashrightarrow \mathcal{F}(\Gamma_{2d-8}), \qquad V_d/G \dashrightarrow \mathcal{F}(\Gamma_{2d-9}),$$

where $\mathcal{F}(\Gamma_n)$ is the modular variety defined by Γ_n. One has $\Gamma_n = O(L_n)_0$ for $1 \leq n \leq 6$, and for $n = 7, 8$ one has $\Gamma_n \supset O(L_n)_0$ with $\Gamma_n/O(L_n)_0 \simeq \mathfrak{S}_{n-5}$.

Here *birational period map* means that I associate a Hodge structure for each general point set in U_d or V_d, and this assignment defines a rational map from the geometric quotient to the arithmetic quotient which is proved to be birational.

Theorem 5.1 for $d = 5, 6$ recovers a result of Matsumoto-Sasaki-Yoshida [9]. They first found period maps of orthogonal type for U_6, V_6, U_5, V_5 by using six lines on $\mathbb{P}^2$. However, our period maps for $d = 5, 6$ differ from those of MSY. The differences are given by Cremona transformations on the configuration spaces. For example, for U_6 the Cremona transformation is of order 12.

The seven period maps except for U_8 can be obtained from the period map for U_8 by degeneration. Indeed, V_d is a component of the boundary of U_d, and U_d in turn is embedded in the boundary of V_{d+1}; on the modular variety side, $\mathcal{F}(\Gamma_n)$ may be embedded in $\mathcal{F}(\Gamma_{n+1})$ as a component of the Heegner divisor for the (-2)-vectors. Then a period map in Theorem 5.1 can be obtained from the one higher dimensional period map by specializing to the boundary. Thus Theorem 5.1 extends the work of MSY after modification by Cremona transformation. The whole resolution of the birational maps is a future task.

Kondō, Dolgachev, and van Geemen described the configuration space U_d/G for $5 \leq d \leq 7$ as an arithmetic quotient of a complex ball (see the lecture [4]). It is also classically known that U_7/G can be expressed as a Siegel modular variety. Thus the space U_d/G for $5 \leq d \leq 7$ admits the structure of an arithmetic quotient in more than one way. In view of the relation of U_d/G with the moduli of del Pezzo surfaces, it would be interesting to describe the induced rational action of the Weyl group on the arithmetic quotient $\mathcal{F}(\Gamma_{2d-8})$.

6. Proof of rationality

Now I explain the proof of Theorem 3.2. Rationality is in general far more delicate than unirationality. Basically I have to give ad-hoc proof for one moduli space by one moduli space. However, there is a common strategy for most of the moduli spaces: we describe the moduli space as a rational quotient of an algebraic variety by an algebraic group, and then prove the rationality of the rational quotient by using techniques in invariant theory. More precisely,

(1) We find a parameter space U of certain (singular) curves lying on some rational surfaces such as $\mathbb{P}^2$, Hirzebruch surface, or del Pezzo surface. On U an algebraic group G acts. For example, G is the the automorphism group of the rational surfaces or some PGL_N.
(2) We construct a period map $p : U \to \mathcal{M}_{(r,a,\delta)}$ by taking the double covers of the rational surfaces branched along the curves in U, and then taking the minimal resolutions of the double cover. The period map p is shown to be G-invariant, so it descends to a rational map $\mathcal{P} : U/G \dashrightarrow \mathcal{M}_{(r,a,\delta)}$ from a rational quotient U/G of U by G.
(3) We prove that $\mathcal{P}$ is birational. The ingredients of the proof are the equality $\dim(U/G) = \dim\mathcal{M}_{(r,a,\delta)}$, the strong Torelli theorem for $K3$ surfaces, and calculation of the order of some finite orthogonal group.
(4) Finally we prove the rationality of U/G by applying various techniques in invariant theory, as explained in [2].

The final step is most essential and most ad-hoc. This proof brings some by-product.

Firstly, that we found a birational period map $\mathcal{P} : U/G \dashrightarrow \mathcal{M}_{(r,a,\delta)}$ means that we have a canonical construction of a general member of $\mathcal{M}_{(r,a,\delta)}$. This has applications to the geometry of 2-elementary $K3$ surfaces. Also, through the quotient U/G, we find that some of $\mathcal{M}_{(r,a,\delta)}$ are related to certain moduli of curves (via the fixed curve of involution). A typical example of this kind is the birational equivalence $\mathcal{M}_{(5,5,1)} \sim \mathcal{M}_6$ used in [11]. Here are some other examples.

- $\mathcal{M}_{(16,2,1)} \sim$ universal genus 2 curve $\mathcal{M}_{2,1}$.
- $\mathcal{M}_{(10,4,1)} \sim$ universal genus 4 curve $\mathcal{M}_{4,1}$.
- $\mathcal{M}_{(5,3,1)} \sim$ moduli of genus 7 trigonal curves.
- $\mathcal{M}_{(4,2,1)} \sim$ moduli of genus 8 trigonal curves with scroll invariant 2.

For more details and more examples, see [8]. The rationality of $\mathcal{M}_{2,1}$ and $\mathcal{M}_{4,1}$ were established by Dolgachev [2] and Catanese [1] respectively, so the rationality of $\mathcal{M}_{(16,2,1)}$ and $\mathcal{M}_{(10,4,1)}$ are reduced to these known rationality. On the other hand, the rationality of the latter two moduli of trigonal curves were unknown, hence we obtain as by-products the rationality of those moduli spaces. We note that for higher g the main component C^g of the fixed curve X^ι is a rather "special" curve: more quantitatively, C^g has Clifford index ≤ 1 when $k \geq 1$, and has Clifford index ≤ 2 in general.

Another by-product of the birational equivalence $U/G \sim \mathcal{M}_{(r,a,\delta)}$ is that we have two compactifications concerning $\mathcal{M}_{(r,a,\delta)}$: the one is the Baily-Borel compactification of $\mathcal{M}_{(r,a,\delta)}$ through its structure as an arithmetic quotient; the other is the GIT compactification of an open set of $\mathcal{M}_{(r,a,\delta)}$ through its structure as a quotient of (an open set of) U by G. Sometimes it might be interesting to compare these two kinds of compactifications.

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Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mail address: `sma@ms.u-tokyo.ac.jp`