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# Birational aspects of moduli of stable sheaves on a surface
## – K-flips and beyond them –

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The topic of this article is some analogy between the wall-crossing problem of moduli schemes of stable sheaves on a surface, and the minimal model program of higher-dimensional varieties. We hope to apply theory of MMP for moduli scheme $M(H)$ of $H$-semistable sheaves, and to consequently grasp its global structure. Here we explain Main result concerning analogy of MMP seen in wall-crossing problem of $M(H)$ when Kodaira dimension of $X$ is positive. We also glance at some finiteness result observed in wall-crossing problem when Kodaira dimension of $X$ is zero.

**Background:** Let $X$ be a non-singular projective surface over $\mathbb{C}$, and $H$ an ample line bundle on $X$. Denote by $M(H)$ (resp. $M^s(H)$) the coarse moduli scheme of $H$-semistable (resp. $H$-stable) sheaves on $X$ with Chern class $\alpha = (r > 1, c_1, c_2) \in \mathbb{N} \times \mathrm{Pic}(X) \times \mathbb{Z}$. When $X$ is a $K3$ surface and $\alpha$ is primitive, $M(H)$ is complex symplectic variety (Mukai), so its Kodaira dimension is zero. On the other hand, when Kodaira dimension of $X$ is positive, Kodaira dimension of $M(H)$ is not well-known. For example, when $X$ is of general type, we *only* know the following work of J. Li [3]:

Let $X$ be a minimal surface of general type. Fix an ample divisor $H$ and a line bundle $c_1 \in \mathrm{Pic}(X)$. Assume: i) there exists a reduced canonical curve $D$ and ii) $\dim M(H, \alpha = (2, c_1, c_2))$ is even. Then for $c_2 \gg 0$ the moduli space $M(H, \alpha)$ is a normal irreducible variety of general type.

However this result is important, conditions i) and ii) in there are not weak. It seems difficult to remove the condition that $p_g(X) = h^0(K_X) > 0$ from the proof of this work. Under this present situation, we shall try to know birational structure of $M(H)$, especially its Kodaira dimension, when Kodaira dimension of $X$ is positive.

**Main theorem:** Let $H_-$ and $H_+$ be $\alpha$-generic polarizations such that just one $\alpha$-wall $W$ separates them For $a \in [0, 1]$ one can define the $a$-semistability of sheaves on $X$ and the coarse moduli scheme $M(a)$ (resp. $M^s(a)$) of rank-two $a$-semistable (resp. $a$-stable) sheaves with Chern classes $\alpha$ in such a way that $M(\epsilon) = M(H_+)$ and $M(1-\epsilon) = M(H_-)$ if $\epsilon > 0$ is sufficiently small $M(a)$ is projective over $\mathbb{C}$. Let $a_- < a_+$ be minichambers separated by only one miniwall $a_0$, and denote $M_+ = M(a_+)$, $M_- = M(a_-)$ and $M_0 = M(a_0)$. There are natural morphisms $\phi_- : M_- \to M_0$ and $\phi_+ : M_+ \to M_0$ (Ellingsrud-Göttshce, Matsuki-Wentworth).

One may say they are morphisms of moduli schemes coming from wall-crossing methods. Let $\phi_- : V_- \to V_0$ be a birational projective morphism such that (1) $V_-$ is normal, (2) $-K_{V_-}$ is $\mathbb{Q}$-Cartier and $\phi_-$-ample, (3) the codimension of the exceptional set $\mathrm{Exc}(\phi_-)$ is more than 1, and (4) the relative Picard number $\rho(V_-/V_0)$ of $\phi_-$ is 1. After the theory of minimal model program, we say a birational projective morphism $\phi_+ : V_+ \to V_0$ is a *K-flip* of $\phi_- : V_- \to V_0$ if (1) $V_+$ is normal, (2) $K_{V_+}$ is $\mathbb{Q}$-Cartier and $\phi_+$-ample, (3) the codimension of the exceptional set $\mathrm{Exc}(\phi_+)$ is more than 1, and (4) the relative Picard number $\rho(V_+/V_0)$ of $\phi_+$ is 1.

**Theorem 0.1** ([6], [7]). *Fix a closed, finite, rational polyhedral cone $\mathcal{S} \subset \overline{\mathrm{Amp}}(X)$ such that $\mathcal{S} \cap \partial\overline{\mathrm{Amp}}(X) \subset \mathbb{R}_{\geq 0} \cdot K_X$. If $c_2$ is sufficiently large with respect to $c_1$ and $\mathcal{S}$, then for any $\alpha$-generic polarizations $H_-$ and $H_+$ in $\mathcal{S}$ separated by just one $\alpha$-wall $W$, and for any adjacent minichambers $a_- < a_+$ separated by a miniwall $a_0$ we have (i) and (ii) when $r = 2$, and (iii) for every $r \geq 2$.*
*(i) $M_\pm$ are normal and $\mathbb{Q}$-factorial, $K_{M_\pm}$ are Cartier, $M^s_\pm$ are l.c.i., and $M_-$ and $M_+$ are isomorphic in codimension 1.*
*(ii) Suppose $K_X$ does not lie in the $\alpha$-wall, and that $K_X$ and $H_+$ lie in the same connected components of $\mathrm{NS}(X)_{\mathbb{R}} \setminus W$. Then $\rho(M_-/M_0) = 1$ and $\phi_+ : M_+ \to M_0$ is a K-flip of $\phi_- : M_- \to M_0$. This morphism $\phi_+$ (resp. $\phi_-$) is the contraction of an extremal ray of $\overline{\mathrm{NE}}(M_+)$ (resp. $\overline{\mathrm{NE}}(M_-)$), which is described in moduli theory.*
*(iii) Suppose $X$ is minimal and $\kappa(X) > 0$, which means $K_X$ is not numerically equivalent to $0$ and contained in $\overline{\mathrm{Amp}}(X)$. Then there is a polarization, say $H_X$, contained in $\mathcal{S}$ such that no $\alpha$-wall separates $H_X$ and $K_X$, and the canonical divisor of $M(H_X)$ is nef.*

By this theorem, when $H$ moves in $\mathrm{Amp}(X)$ and gets closer to $K_X$ and get to $H_X$, we can find the following famous process in minimal model program for $M(H)$, in wall-crossing problem of moduli theory: Start from a normal variety $V$ $\Rightarrow$ If $K_V$ is not nef, find some $K$-negative extremal ray and contract it ($V \to V_0$) $\Rightarrow$ If contraction is small, find its $K$-flip ($V_0 \leftarrow V_+$) $\Rightarrow$ By repeating it finitely, get to a minimal model $\tilde{V}$ of $V$, i.e. (i) $\tilde{V}$ is birational to $V$, (ii) $K_{\tilde{V}}$ is $\mathbb{Q}$-Cartier and nef, and (iii) $\tilde{V}$ admits only terminal singularities.

Thus, $M(H_X)$ is a moduli-theoretic analogue of minimal model of $M(H)$, and so $M(H_X)$ seems to be suitable to the aim of studying birational structure of $M(H)$, although we do not know whether its singularities are terminal or not.

**Future subjects:** The only general signpost for surfaces of general type is concerned with pluricanonical class $mK_X$. So we can hope that it should be meaningful to watch pluricanonical class/map of $M(H_X)$ in order to understand it.

When $K_X$ is ample, J. Li's work ([2]) implies that pluricanonical map of $M(K_X)$ is highly related with its Uhlenbeck compactification. Our main theorem suggests that pluricanonical map of $M(H_X)$ is worth of attention, but the problem is that bad singularities can exist. We have almost NO methodology to inquire whether singularities of moduli of sheaves are mild or not, when $X$ is of general type. Therefore we need to consider the following question:

**Question 0.2.** Does $M(H_X)$ admit only mild singularities if $2rc_2-(r-1)c_1^2 \gg 0$? For example, can we set out this question by considering the following?

- Estimate the degree of lowest (and highest, if any) term of $f_i$.
- How far (or different) are $f_1, \dots, f_b$ from one another? (Formulate this problem mathematically.)

Let $k[[x_1, \dots, x_{d+b}]]/(f_1, \dots, f_b)$ be the completion of $M(H_X)$ at $E$. Here $d+b = \dim \mathrm{Ext}^1(E,E)^0$ and $b = \dim \mathrm{Ext}^2(E,E)^0$, where trailing zero means the kernel of trace map $\mathrm{Ext}^i(E,E) \to H^i(\mathcal{O}_X)$. Remark that $M(H)$ is normal and $M^s(H)$ is l.c.i.. if $2rc_2-(r-1)c_1^2 \gg 0$ (Gieseker-Li). Kuranishi theory relates moduli theory with the way to determine coefficients in $f_i(x)$. For example, Laudal ([1]) states that the degree-two term agrees with (trace-zero version of) the dual map

$$\mathrm{Ext}^2(E,E)^\vee \longrightarrow \mathrm{Sym}^2(\mathrm{Ext}^1(E,E))^\vee$$

of the cup-product map

$$\mathrm{Ext}^1(E,E) \otimes \mathrm{Ext}^1(E,E) \longrightarrow \mathrm{Ext}^2(E,E).$$

Thus the degree-two part of $f_i$ is relatively familiar with us. In [1], also higher-degree part is described by using terms of $\mathrm{Ext}^*$ and methods called Massey product. We shall attack Question 0.2, for example, by using these facts.

**Finiteness result when $\kappa(X)=0$:** When $K_X$ is numerically equivalent to 0, also the canonical class of $M(H)$ is numerically equivalent to zero. Then birational maps coming from wall-crossing are moduli-theoretic analogues of $K$-flops, and $M(H)$ is a moduli-theoretic analogue of minimal model of moduli of sheaves for any $\alpha$-generic polarization $H$. Matsuki-Wentworth [4] pointed out an example of an Abelian surface and a class $\alpha$ such that infinitely many $\alpha$-walls exists in $\mathrm{Amp}(X)$. Then it seems that infinitely many minimal model of moduli of sheaves exist on first glance, but the following proposition says that we practically get only finitely many birational varieties by wall-crossing method when $K_X \equiv 0$.

**Proposition 0.3** ([5])**.** *A line bundle $L$ on $X$ and $\tau \in \mathrm{Aut}(X)$ respectively induce $\otimes L$ and $\tau_* \in \mathrm{Aut}(\mathrm{Coh}(X))$, that is the automorphism group of $\mathrm{Coh}(X)$. Then let $\mathrm{Aut}'(X, c_1)$ be the subgroup of $\mathrm{Aut}(\mathrm{Coh}(X))$ generated by $\otimes L \circ \tau_*$ which satisfies*

$c_1(L \otimes \tau_*(E)) = c_1$ *in* $\mathrm{NS}(X)$ *for a sheaf* $E$ *on* $X$ *with rank* $r$ *and* $c_1(E) = c_1$. $\mathrm{Aut}'(X, c_1)$ *naturally acts on*

$$\mathcal{F}(\alpha) = \left\{ M(H, \alpha) \;\middle|\; H\text{: } \alpha\text{-generic polarization on } X \right\}.$$

*Suppose that* $K_X$ *is numerically equivalent to zero, that is,* $X$ *is minimal and* $\kappa(X) = 0$. *When* $r > 0$, *the set* $\mathcal{F}(\alpha) / \mathrm{Aut}'(X, c_1)$ *is finite.*

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