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ON THE GIT STABILITY OF POLARIZED VARIETIES -A SURVEY-

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Abstract. "When a polarized variety is (GIT) stable?" is concretely studied, under the background of the existence of the "canonical metric" and the problem of moduli construction. This is also a survey to the algebraic aspect of the field, which lies between algebraic geometry and differential geometry, written as the proceeding of the Kinosaki algebraic geometry conference 2010.

1. Introduction

We start from differential geometric background. As Calabi conjectured in 1950s [Cal54], now it is widely recognized that

Theorem 1.1. *Kähler-Einstein metric exists on*
(i)([Aub76], [Yau78]) *a projective manifold* X *with* $c_1(X) < 0$*,*
(ii)([Yau78]) (*or*) *a compact Kähler manifold* $c_1(X) = 0$ *with any fixed Kähler class.*

The Kähler-Einstein metric is a kind of "canonical" Kähler metric which has a constant scalar curvature, generalizing a Kähler metric on a compact Riemann surface obtained by uniformization. For example, if the automorphisms fixing the Kähler class form a discrete group, they are isometry with respect to the Kähler-Einstein metric, due to the uniqueness modulo the connected component of the automorphism group. In this sense of "canonicity", we might say that this is opening a way connecting differential geometry to algebraic geometry. Such a metric could exist only on complex manifolds of type (i), (ii) or Fano manifolds. However, it is known for decades that a Fano manifold does *not* necessarily have a Kähler-Einstein metric due to the obstructions of Matsushima [Mat57] and Futaki [Fut83].

On the other hand, it seems that Yau has further imagined that the its existence should be equivalent to a sort of GIT stability in general (cf. e. g. [Yau90]), from his estimates on "approximating solutions" (made by the continuity method on the complex Monge-Ampère equation). It was known as a folklore status, made to mathematically rigorous conjecture firstly by Tian [Tia97], reformulated and generalized by Donaldson [Don02].

Fujiki [Fuj90] and Donaldson [Don97], [Don04] introduced the following (infinite dimensional) *moment map* picture for explanation, which treats more general "canonical" metric; that is a Kähler metric with constant scalar curvature (cscK) (or more general "extremal Kähler metric"). We note that if we fix a Kähler class, which is proportional to the first Chern class $c_1(X)$, it is equivalent to the Kähler-Einstein metric.

Let us fix a compact connected differential manifold and a symplectic form (X, ω). Consider the space $\mathcal{C}$ of all compatible complex structures, acted by a group D of symplectomorphisms. Then, the (normalized) scalar curvature would be a moment map from $\mathcal{C}$ with a certain natural symplectic structure, with a certain subgroup D_P of D. The equivalence relation between abstractly isomorphic polarized complex manifolds can be thought of as a sort of the action of "complexification" of D_P (although the complexification is not defined). Therefore, the following is a natural conjecture as an infinite dimensional example of a version of Kempf-Ness' figure of the correspondence of GIT quotient and the symplectic quotient of the zero set of the moment map [KN79], [Kir84].

Philosophy 1.2. *For a polarized manifold (X, L), cscK metric with Kähler class $c_1(L)$ exists if and only if it is GIT stable in some sense.*

The historically original notion of GIT stability for polarized varieties is *asymptotic* (*Hilbert or Chow*) *stability*, which was studied intensively by Mumford, Gieseker in 1960s-70s. We remark that asymptotic Chow stability and asymptotic Hilbert stability is actually equivalent, due to [Mab08] (cf. [Od09a] for a simplified proof in one page). For these notions, the following is proved.

Theorem 1.3 ([Don01])**.** *(X, L) with cscK metric and discrete automorphism group is asymptotically stable.*

The proof is based on the well known fact that, for an embedded projective manifold $X \subset \mathbb{P}$, the existence of *balanced metric* is equivalent to Chow stability, and regard cscK as their limit. The converse of Theorem 1.3 now seems to be known that it is false by Julius Ross and Julien Keller [JK].

To make the philosophy 1.2 in precise form, Tian introduced the notion of K-stability in [Tia97] which is reformulated by Donaldson in [Don02] and the following conjecture is formulated.

Conjecture 1.4 (cf. [Yau90], [Tia97], [Don02]). *Let (X, L) be a polarized projective manifold. X has a Kähler metric with constant scalar curvature (cscK metric) with Kähler class $c_1(L)$ if and only if (X, L) is K-polystable.*

One direction of Conjecture 1.4, i. e. that the existence of cscK implies K-polystability, is settled by [Tia97], [Don05], [CT08], [Stp08], and in full generality by [Mab08] and [Mab09].

Roughly speaking, the asymptotic stability of (X, L) is just the stability of embedded $X \subset \mathbb{P}(H^0(X, L^{\otimes a})$ for $a \gg 0$. On the other hand, K-stability is defined as positivity of all Donaldson-Futaki invariants which corresponds to each *test configuration*, the "geometrization" of one parameter subgroup of $\mathrm{GL}(H^0(X, L^{\otimes a}))$ for $a \gg 0$. The Donaldson-Futaki invariant is a kind of GIT-weight. A formula of Donaldson-Futaki invariant is firstly obtained by Xiaowei Wang in [Wan08].

The author's work presented in the talk is summarised as

(i) (independently but lator) obtained the following **different version of the formula** of the Donaldson-Futaki invariants for *semi*-test configurations of *specific* type, which is **enough** for showing K-stability (resp. K-semistability). The point is that it is applicable, and described in terms of closed subschemes. *This is an extension of Ross-Thomas' theory.*

(ii) as an application of the formula, established **the effects of singularities**, with the language of **discrepancy**, introduced along the minimal model program (MMP). The proof itself uses the existence of certain *birational model*. This extends the work of Mumford and Shah in 1970s, for curves and surfaces.

(iii) as other applications of the formula, we established **"algebraically counterparts"** of the results of the existence of Kähler-Einstein metrics by Aubin, Yau and Tian. (For Tian's, I coworked with Yuji Sano.) We can admit *mild singularities.*

Now, we recall the definitions of the stability notions. We note that "$*$-unstable" means "not $*$-*semi*stable". The readers who are not interested in technical details, might be able to skip the definitions. Please consult [Don02, Chapter 2, especially 2.3], [RT07, Section 3] or [Od09a] for more related informations.

First, we review the definition of asymptotic stabilities.

Definition 1.5. A polarized scheme (X, L) is said to be *asymptotically Chow stable* (resp. *asymptotically Hilbert stable*, *asymptotically Chow semistable*, *asymptotically Hilbert semistable*), if for an arbitrary $m \gg 0$, $\phi_m(X) \subset \mathbb{P}(H^0(X, L^{\otimes m}))$ is Chow stable (resp. Hilbert stable, Chow semistable, Hilbert semistable), where ϕ_m is the closed immersion defined by the complete linear system $|L^{\otimes m}|$.

To define the K-stability, we review the concept of test configuration following Donaldson [Don02]. Our notation (and even expression) almost follows [RT07], so we refer to it for details.

Definition 1.6. A *test configuration* (resp. *semi test configuration*) for a polarized scheme (X, L) is a polarized scheme $(\mathcal{X}, \mathcal{L})$ with:

(i) a $\mathbb{G}_m$ action on $(\mathcal{X}, \mathcal{L})$
(ii) a proper flat morphism $\alpha\colon \mathcal{X} \to \mathbb{A}^1$

such that α is $\mathbb{G}_m$-equivariant for the usual action on $\mathbb{A}^1$:

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \\ (t, x) & \longmapsto & tx, \end{array}$$

$\mathcal{L}$ is relatively ample (resp. relatively semi ample), and $(\mathcal{X}, \mathcal{L})|_{\alpha^{-1}(\mathbb{A}^1 - \{0\})}$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, L^r) \times (\mathbb{A}^1 - \{0\})$ for some positive integer r, called *exponent*, with the natural action of $\mathbb{G}_m$ on the latter and the trivial action on the former.

Proposition 1.7 ([RT07, Proposition 3.7]). *In the above situation, a one-parameter subgroup of* $GL(H^0(X, L^{\otimes r}))$ *is equivalent to the data of a test configuration with exponent* r *of* (X, L) *for* $r \gg 0$.

We will call the test confinguration which corresponds to a one parameter subgroup, called the *DeConcini-Procesi family.* (Its curve case already appears in [Mum65, Chapter 4 §6].) Therefore, the test configuration can be regarded as *geometrization* of one-parameter subgroup. This is a quite essential point for our study, as in Ross and Thomas' slope theory [RT06], [RT07].

Let (X, L) be an n-dimensional polarized variety. A *test configuration* (resp. a *semi test configuration*) for (X, L) is a polarize scheme $(\mathcal{X}, \mathcal{L})$ with a $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{L})$ and a proper flat morphism $\Pi\colon \mathcal{X} \to \mathbb{A}^1$ such that (i) Π is $\mathbb{G}_m$-equivariant for the multiplicative action of $\mathbb{G}_m$ on $\mathbb{A}^1$, (ii) $\mathcal{L}$ is relatively ample (resp. relatively semi-ample), and (iii) $(\mathcal{X}, \mathcal{L})\ |_{\Pi^{-1}(\mathbb{A}^1 - \{0\})}$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, L^{\otimes r}) \times (\mathbb{A}^1 - \{0\})$ for some positive integer r. If $\mathcal{X} \simeq X \times \mathbb{A}^1$, we call $(\mathcal{X}, \mathcal{L})$ a *product test configuration.* Moreover, if $\mathbb{G}_m$ acts trivially, we call it a trivial test configuration. Let $P(k) := \dim H^0(X, L^{\otimes k})$, which is a polynomial in k of degree n due to Riemann-Roch theorem. Since the $\mathbb{G}_m$-action preserves the central fibre $\mathcal{X}_0$ of $\mathcal{X}$, $\mathbb{G}_m$ acts also on $H^0(\mathcal{X}_0, \mathcal{L}^{\otimes K} |_{\mathcal{X}_0})$, where $K \in \mathbb{Z}_{>0}$. Let $w(Kr)$ be the weight of the induced action on the highest exterior power of $H^0(\mathcal{X}_0, \mathcal{L}^{\otimes K} |_{\mathcal{X}_0})$, which is a polynomial of K of degree $n+1$ due to the Mumford's droll Lemma (cf. [Mum77, Lemma 2.14]) and Riemann-Roch theorem. Here, the *total weight* of an action of $\mathbb{G}_m$ on some finite-dimensional vector space is defined as the sum of all weights, where the *weights* mean the exponents of eigenvalues which should be powers of $t \in \mathbb{A}^1$. Let us take $rP(r)$-th power and SL-normalize the action of $\mathbb{G}_m$ on $(\Pi_*\mathcal{L})|_{\{0\}}$, then the corresponding normalized weight on $(\Pi_*\mathcal{L}^{\otimes K})|_{\{0\}}$ is $\tilde{w}_{r,Kr} := w(k)rP(r) - w(r)kP(k)$, where $k := Kr$. It is a polynomial of form $\sum_{i=0}^{n+1} e_i(r)k^i$ of degree $n+1$ in k for $k \gg 0$, with coefficients which are also polynomial of degree $n+1$ in r for $r \gg 0$: $e_i(r) = \sum_{j=0}^{n+1} e_{i,j} r^j$ for $r \gg 0$. Since the weight is normalized, $e_{n+1,n+1} = 0$. The coefficient $e_{n+1,n}$ is called the *Donaldson-Futaki invariant* of the test configuration, which we denote by $\mathrm{DF}(\mathcal{X}, \mathcal{L})$. For an arbitrary *semi* test configuration $(\mathcal{X}, \mathcal{L})$ of order r, we can also define the Donaldson-Futaki invariant as well by setting $w(Kr)$ as the total weight of the induced action on $H^0(\mathcal{X}, \mathcal{L}^{\otimes K})/tH^0(\mathcal{X}, \mathcal{L}^{\otimes K})$ (cf. [RT07]). We say that (X, L) is K-stable (resp. K-semistable) if and only if $\mathrm{DF} > 0$ (resp. $\mathrm{DF} \geq 0$) for any non-trivial test configuration. We also say that (X, L) is K-polystable if and only if $\mathrm{DF} \geq 0$ for any non-trivial test configuration and $\mathrm{DF} = 0$ only if a test configuration is a product test configuration.

We make a small remark on an extension of the framework above. If we take a test configuration (resp. semi test configuration) $(\mathcal{X}, \mathcal{L})$, we can think of a new test configuration (resp. semi test configuration) $(\mathcal{X}, \mathcal{L}^{\otimes a})$ with $a \in \mathbb{Z}_{>0}$. From the definition of Donaldson-Futaki invariant above, we easily see that $\mathrm{DF}((\mathcal{X}, \mathcal{L}^{\otimes a})) = a^n \mathrm{DF}((\mathcal{X}, \mathcal{L}))$. Therefore, we can define K-stability (also K-polystability and K-semistability) of a pair (X, L) of a projective scheme X and an ample **$\mathbb{Q}$-line** bundle L.

We refer to [Od09a], [Od09b], [Od10], and [OS10] for the details and proofs of our works. We should confess here that this short survey, titled as like that, is still biased and focused on the viewpoint of the author's study done so far.

2. A generalization of Ross-Thomas' theory

The following explicit algebro-geometric formula of Donaldson-Futaki invariants is given by Xiaowei Wang.

Theorem 2.1. ([Wan08, Proposition 19]) *For any* (*ample*) *test configuration* $(\mathcal{X}, \mathcal{M})$ *of a polarized variety* (X, L), *if we denote its natural compactification as* $(\bar{\mathcal{X}}, \bar{\mathcal{M}})$, *the corresponding Donaldson-Futaki invariant is the following* ;

$$DF(\mathcal{X}, \mathcal{M}) = \frac{1}{2(n!)((n+1)!)}\{-n(L^{n-1}.K_X)(\bar{\mathcal{M}}^{n+1})+(n+1)(L^n)(\bar{\mathcal{M}}^n.K_{\bar{\mathcal{X}}/\mathbb{P}^1})\}.$$

Here, $K_{\bar{\mathcal{X}}/\mathbb{P}^1}$ *means the divisor* $K_{\bar{\mathcal{X}}} - f^*K_{\mathbb{P}^1}$ *with the projection* $f\colon \bar{\mathcal{X}} \to \mathbb{P}^1$.

The definition of "natural compactification" follows the trivialization of $\mathcal{X} \setminus \Pi^{-1}(0)$. Please refer to [Wan08] for the detail and the proof. From this formula, we can say that, at least in principle, the original Futaki invariants or the Futaki charactor of X can be recovered in algebro-geometric way, by using the fiber bundles over $\mathbb{P}^1$ with fiber isomorphic to X.

Our version of the formula is the following, which is a little lengthy.

Theorem 2.2. ([Od09a]) *For any flag ideal* $\mathcal{J} \subset \mathcal{O}_{X\times\mathbb{A}^1}$ (*cf.* [Od09a]), *consider the "semi" test configuration* $(Bl_{\mathcal{J}}(X \times \mathbb{A}^1) =: \mathcal{B}, \mathcal{L}(-E))$ *of blow up type with* (*relatively*) *"semi"ample* $\mathcal{L}(-E)$ *where* $\Pi^{-1}\mathcal{J} = \mathcal{O}_{\mathcal{B}}(-E)$. *Here,* $\Pi\colon \mathcal{B} \to X \times \mathbb{A}^1$ *is the blowing up morphism. Let us write its natural compactification as* $(Bl_{\mathcal{J}}(X \times \mathbb{P}^1) =: \bar{\mathcal{B}}, \overline{\mathcal{L}(-E)})$ *and let* p_i $(i = 1, 2)$ *be the projection from* $X \times \mathbb{P}^1$. *Then, if* $\mathcal{B}$ *is Gorenstein in codimension* 1, *the Donaldson-Futaki invariant of the semi test configuration can be expanded in the following way*;

$$\begin{aligned}
&2(n!)((n+1)!)DF(\mathcal{B}, \mathcal{L}(-E)) \\
&= -n(L^{n-1}.K_X)(\overline{(\mathcal{L}-E)}^{n+1}) + (n+1)(L^n)(\overline{(\mathcal{L}(-E))}^{n}.K_{\bar{\mathcal{B}}/\mathbb{P}^1}) \\
&= -n(L^{n-1}.K_X)(\overline{(\mathcal{L}-E)}^{n+1}) + (n+1)(L^n)(\overline{(\mathcal{L}(-E))}^{n}.\Pi^*(p_1^*K_X)) \\
&+ (n+1)(L^n)(\overline{(\mathcal{L}(-E))}^{n}.K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1}).
\end{aligned}$$

Here, $K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1}$ *means* $K_{\bar{\mathcal{B}}} - \Pi^*K_{X\times\mathbb{P}^1}$.

The point is that the formula seems to be more applicable, and also expressed in viewpoints of the *closed subscheme* of X or $X \times \mathbb{A}^1$, yielding the concept of "destabilizing subscheme", with an analogy to the theory on vector bundles. We will explain its effectivity lator on. A *flag ideal* $\mathcal{J} \subset \mathcal{O}_{X\times\mathbb{A}^1}$ means a coherent ideal of the form

$$\mathcal{J} = I_0 + I_1t + I_2t^2 + \cdots + I_{N-1}t^{N-1} + (t^N),$$

where $I_0 \subset I_1 \subset \cdots I_{N-1} \subset \mathcal{O}_X$ is a sequence of coherent ideals of X (cf. [Od10, Definition 3.1]). The formula (ii) is useful by its form. Let us recall that we named the former line (two terms) the "canonical divisor part" which is the intersection numbers with canonical divisor K_X or its pull back and the latter line (one term) the "discrepancy term" which reflects the singularities of X. Namely the canonical divisor part is defined as

$$-n(L^{n-1}.K_X)(\overline{(\mathcal{L}-E)}^{n+1}) + (n+1)(L^n)(\overline{(\mathcal{L}(-E))}^n.\Pi^*(p_1^*K_X)),$$

which we denote $DF_{cdp}(\mathcal{B}, \mathcal{L}(-E))$ and the discrepancy term is defined as

$$DF_{dt}(\mathcal{B}, \mathcal{L}(-E)) := (n+1)(L^n)(\overline{(\mathcal{L}(-E))}^n.K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1}).$$

We note that this discrepancy term is non-negative if X has only semi-log-canonical singularities (cf. [Od09a]). That is one of the points for the applications.

Remark 2.3. The theory of Ross-Thomas' "slope" treats $N = 1$ case, i.e. when the ideal is of the form $\mathcal{J} = I + (t)$, of the above formula. In that sense, our formula is the extension of their theory. Their idea is to describe the corresponding Donaldson-Futaki invariant as $\mu(X, L) - \mu_c(I, (X, L))$, as an analogy to the theory of slope for vector bundles by Mumford and Takemoto. Therefore, in particular, they describe the invariants in terms of cohomological quantities of X and their ideal and its blow up (not that of $X \times \mathbb{A}^1$ or its blow up).

We remark that, however, it is later observed [PR07] that all the Donaldson-Futaki invariants of 2 points blow up of projective plane of such type is positive (i.e. slope stable), but it is K-**un**stable. Therefore, the slope theory is not enough to check K-(semi)stability. On the other hand, we further note here that for some more general (but *not* all) flag ideals $\mathcal{J}$, not necessarily of the simplest form $\mathcal{J} = I + (t)$, we can also write the corresponding Donaldson-Futaki invariants by using the cohomological quantity of X and their ideals, similarly as in the original Ross-Thomas' slope theory. For example, we can do it for the case of weighted blow up, just by imitating the calculation of Donaldson-Futaki invariants in [RT07].

The following says that it suffices to consider all semi test configurations only type of $(\mathcal{B}, \mathcal{L}(-E))$ in order to show K-stability. We note that this is obtained as an application of [RT07, (proof of) Proposition 5.1, Remark 5.2].

Proposition 2.4 ([Od10, Proposition 2.2])**.** *(X, L) is K-stable if and only if $DF(\mathcal{B}, \mathcal{L}(-E)) > 0$ for all flag ideals and $r \in \mathbb{Z}_{>0}$ such that $\mathcal{B}$ is Gorenstein in codimension 1 and $\mathcal{L}(-E)$ is semi-ample.*

We just explain one quite trivial example of applications. If (X, L) is semi-log-canonical polarized Calabi-Yau varieties, the canonical divisor part vanishes and the discrepancy term is always non-negative for semi-log-canonical X as we noted. Therefore, such (X, L) is K-semistable by our formula ([Od09a]). From now on, we will see some more results obtained by applying the formula.

The author hopes and expects that, since the formula is given explicitly, it would be applied to study more and more classes of polarized varieties, creating stage where algebro-geometric theories on higher dimensional varieties (or singularities) play important roles.

3. K-stability results

As direct applications of the formula 2.2, we obtained

Theorem 3.1 ([Od09a], [Od10])**.** (i) *A semi-log-canonical canonically polarized variety* $(X, \mathcal{O}_X(K_X))$*, is K-stable.*

(ii) *A log-terminal polarized variety* (X, L) *with numerically trivial canonical divisor* K_X *is K-stable.*

(iii) *A semi-log-canonical polarized variety* (X, L) *with numerically trivial canonical divisor* K_X *is K-semistable.*

These results "corresponds" to the existence of Kähler-Einstein metrics on Calabi-Yau manifolds and smooth canonical model by Aubin and Yau 1.1. We note that these yields many *orbifold* counterexamples to the former folklore conjecture that "K-(poly)stability implies asymptotic (poly)stability" which also implies that Theorem 1.3 of Donaldson does not hold for orbifolds ([Od10]).

For Fano case, by taking the relation with Seshadri constants into account, we obtain the following theorem purely algebro-geometrically by using the formula 2.2. This is a joint work with Yuji Sano (Kyushu university).

Theorem 3.2 ([OS10])**.** *Let* X *be a* (*log-canonical*) $\mathbb{Q}$*-Fano variety with* $\dim(X) = n$ *and suppose that* $\mathrm{lct}(X) > \frac{n}{n+1}$ (*resp.* $\mathrm{lct}(X) \geq \frac{n}{n+1}$)*. Then,* $(X, \mathcal{O}_X(-K_X))$ *is K-stable* (*resp. K-semistable*)*.*

Here, $\mathrm{lct}(X)$ is the *global log canonical threshold* and defined in the following way.

$$\mathrm{lct}(X) = \inf_{D \equiv -K_X} \mathrm{lct}(X, D),$$

where D is an effective $\mathbb{Q}$ divisor, which is numerically equivalent to $-K_X$. We note that there are G-equivariant version of statements in our paper [OS10].

This theorem "corresponds" to the existence of Kähler-Einstein metrics on Fano manifolds under the condition of α-invariants, which is analytically defined but equal to the global log canonical threshold, due to Tian in [Tia87]. His original proof depends on the a priori estimates on the approximation of solutions obtained by the continuity methods, as in Yau's argument [Yau78].

4. Effects of singularities

Mumford [Mum77] and Shah [Sha81], [Sha86] observed that "if there is some **bad** singularities, a polarized variety is **unstable**" and make concrete analysis on 1 and 2 dimensional cases. We generalize it in the following (best possible) form.

Theorem 4.1 ([Od09b], [Od10]). *If (X, L) is K-semistable, X has only normal crossing singularities in codimension 1. Furtheremore, X only admit* ***semi-log-canonical singularities*** *for* $\dim(X) \leq 3$.

We note that K-semistability is almost the weakest stability notions among all introduced (cf. [Od10]). The notion of semi-log-canonicity is the higher dimensional generalization of "smooth or ordinary double point" for curves, defined in terms of the "discrepancy" of canonical divisors among the original singularities and its resolution. For an arbitrary dimension, we also have many results stating that "Theorem 4.1 holds under certain condition". Please consult [Od09a] and [Od10] for the details.

The proof is done by constructing the *destabilizing one parameter subgroup* of $\mathrm{GL}(H^0(X, L^{\otimes k}))$ for $k \gg 0$ geometrically by using a *birational model of X* (the relative log canonical model of log resolution). Therefore the difficulty for the full settlement comes from the need to (partially) establish *non-normal* analogue of MMP. We carried out it partially by using technical but fundamental lemmas by Kollár [Kol10, (23), (53)] to glue a normal (log-canonical) variety to a non-normal (semi-log-canonical) variety (cf. [Od10]).

5. K-stability, moduli and the CM line bundle

Originally, the theory of GIT stability is introduced for the purpose of algebro-geometric construction of the **moduli** of curves by Mumford [Mum65]. General type surface case was treated by [Gie77]. The study on higher dimensional case is difficult and is still far from complete settlement. As far as the author knows, it is known for more than two decades (see [She83]) that for dimension higher than 1, there are canonically polarized varieties, which are "good" degenerations which should be on the boundary of compactified moduli but **asymptotically unstable**. After that observation, the GIT approach was abandaned for a while and another way, without direct relation to stability, to construct moduli was developed and recently completed for general type varieties (cf. e. g. [Kov05]). It seems that Professor János Kollár, as one of the big leaders in the construction, is writing a book on this matter (partly available on his webpage as [Kol10]).

In a viewpoint of moduli construction, our Theorem 3.1 (i) and Theorem 4.1 can be summarized as follows.

Theorem 5.1. *For a variety X with $\dim(X) \leq 3$ whose canonical class K_X is ample, the followings are equivalent.*

(i)X *is semi-log-canonical.*

(ii)X *is a member parametrized by the* (*disjoint union of*) *projective moduli recently constructed by many peoples' contribution* (*cf. e. g.* [Kov05])*, without GIT theory.*

(iii)(X, K_X) *is* ***K-stable****.*

(iv)(X, K_X) *is K-semistable.*

Therefore it is quite natural to expect that more generally

"Conjecture" 5.2. *K-polystable polarized varieties form quasi-projective moduli indeed, and the CM line bundle is ample on it.*

The **CM line bundle** is the line bundle on the base scheme of family of polarized varieties, which is generally introduced in [PT06] whose GIT weight just coincides with the Donaldson-Futaki invariants which defines K-stability.

This resuscitate the idea of GIT stability in the moduli theory of higher dimensional varieties, which corresponds to the differential geometric figure pictured by Fujiki [Fuj90] and Donaldson [Don97] for smooth case, as we explained in the introduction. The CM line bundle have a Hermitian metric whose associated first Chern form is the generalization of **Weil-Petersson metric** which should be positive. Essentially this story is realized analytically by Fujiki-Schumacher [FS90]. Our dream above is therefore, to algebrize these figures, admitting (nearly) semi-log-canonical singularities. We note here that, for higher dimensional cases, the definition of moduli functor is also a problem to be solved, because of the subtlety of the singularities (cf. [Kov05]). Therefore, "Conjecture" 5.2 should be formulated in more precise form. We have only some more partial observations toward the establishment of Conjecture 5.2, and are planning to prepare another paper in future.

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