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Triangulated surface, mapping class group and Donaldson-Thomas theory

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Given a triangulation of a surface, a quiver with a potential is defined ([LF09]). Given a quiver with a potential, a 3-dimensional Calabi-Yau triangulated category is defined ([Gin, KY, Kel]). The mapping class group of the original surface acts on the derived category. As a consequence, the associated Donaldson-Thomas theory is "invariant" under the mapping class group action.

1 QP for a triangulated surface

In this section, we explain how to associate a quiver with a potential for a triangulated surface. The reader may refer [LF09] for the full details.

1.1 Ideal triangulations of a surface

Let Σ be a compact connected oriented surface with (possibly non-empty) boundary and M be a finite set of points on Σ, called marked points. We assume that M is non-empty and has at least one point on each connected component of the boundary of Σ. The marked points that lie in the interior of Σ will be called **punctures**, and the set of punctures of (Σ, M) will be denoted P. [1]

We decompose Σ into "triangles" (in the topological sense) so that each edge is either

- a curve (which is called an *arc*) whose endpoints are in M or

[1] We will always assume that (Σ, M) is none of the following:

- a sphere with less than five punctures;
- an unpunctured monogon, digon or triangle;
- a once-punctured monogon.

Here, by a monogon (resp. digon, triangle) we mean a disk with exactly one (resp. two, three) marked point(s) on the boundary.

- a connected component of $\partial\Sigma \backslash M$.

A triangle may contains exactly two arcs (see Figure 1). Such a triangle (and its doubled arc) is said to be *self-folded*.

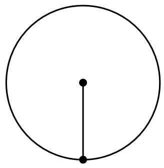

Figure 1: A self-folded triangle

Given a triangulation τ and a (non self-folded) arc i, we can *flip* i to get a new triangulation $f_i(\tau)$ (see Figure 2).

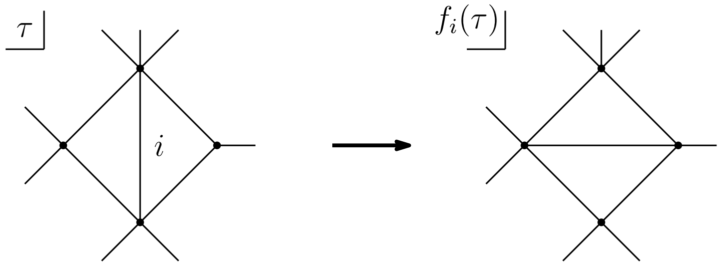

Figure 2: A flip of a triangulation

Theorem 1 ([FST08]). *Any two triangulations are related by a sequence of flips.*

1.2 Quiver for a triangulation

Let τ be a triangulation. We will define a quiver $Q(\tau)$ without loops and 2-cycles whose vertex set I is the set of arcs in τ.

For a (non self-folded[2]) triangle Δ and arcs i and j, we define a skew-symmetric integer matrix B^{Δ} by

$$B^{\Delta}_{i,j} := \begin{cases} 1 & \Delta \text{ has sides } i \text{ and } j, \text{ with } i \text{ following } j \text{ in the clockwise order,} \\ -1 & \text{the same holds, but in the counter-clockwise order,} \\ 0 & \text{otherwise.} \end{cases}$$

[2]We omit the definition of B^{Δ} for a self-folded triangle Δ.

We put

$$B(\tau) := \sum_{\Delta} B^{\Delta}$$

where the sum is taken over all triangles in τ. Let $Q(\tau)$ denote the quiver without loops and 2-cycles associated to the matrix $B(\tau)$.

Theorem 2 ([FST08]). *Given a triangulation τ and its (non self-folded) arc i, we have*

$$Q(f_i(r)) = \mu_i(Q(\tau))$$

where μ_i denote the mutation of the quiver at the vertex i.

1.3 Potential for a triangulation

For a triangle Δ in τ, we define a potential ω_Δ as in Figure 3. For a puncture

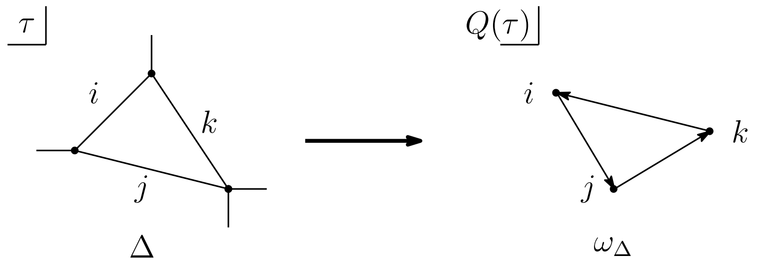

Figure 3: ω_Δ

P in τ, we define a potential ω_P as in Figure 4.

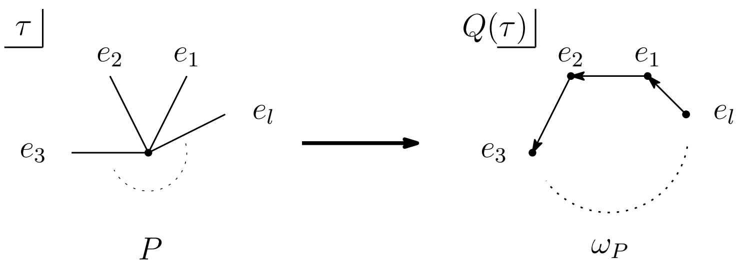

Figure 4: ω_P

Finally, we put

$$\omega(\tau) := \sum_{\Sigma} \omega_\Sigma + \sum_{P} \omega_P.$$

Theorem 3 ([LF09]). *Given a triangulation τ and its (non self-folded) arc i, we have*

$$\omega(f_i(r)) = \mu_i(\omega(\tau))$$

where μ_i denote the mutation of the potential at the vertex i in the sense of [DWZ08].

2 Mapping class group action

2.1 Mapping class group

We define

$$\mathrm{Diffeo}(\Sigma, M) := \{\phi\colon \Sigma \to \Sigma \mid \phi\colon \text{diffeomorphism}, \phi|_M = \mathrm{id}_M\}.$$

Let $\mathrm{Diffeo}(\Sigma, M)_0$ denote the connected component of $\mathrm{Diffeo}(\Sigma, M)$ which contains id_Σ. The quotient

$$\mathrm{MCG}(\Sigma, M) := \mathrm{Diffeo}(\Sigma, M)/\mathrm{Diffeo}(\Sigma, M)_0$$

is called the *mapping class group*.

2.2 Derived category for a triangulation

Let $\Gamma(\tau)$ be Ginzburg's dg algebra associated to the quiver with the potential $(Q(\tau), \omega(\tau))$ and $\mathcal{D}(\tau) = \mathcal{D}\Gamma(\tau)$ be the derived category of right dg-modules over Γ. By the result of Keller ([Kel]), $\Gamma(\tau)$ and $\Gamma(f_i(\tau))$ are equivalent[3].

For a triangulation τ and an element $\phi \in \mathrm{MCG}(\Sigma, M)$, we get another triangulation $\phi(\tau)$. Note that $(Q(\tau), \omega(\tau))$ and $(Q(\phi(\tau)), \omega(\phi(\tau)))$ (and hence $\mathcal{D}(\tau)$ and $\mathcal{D}(\phi(\tau))$) are canonically identified.

By Theorem 1, τ and $\phi(\tau)$ are related by a sequence of flips. Each flips gives a derived equivalence. By composing the derived equivalences, we get a derived equivalence

$$\Psi_\phi\colon \mathcal{D}(\tau) \xrightarrow{\sim} \mathcal{D}(\phi(\tau)) = \mathcal{D}(\tau).$$

Thanks to the result [FST08, Theorem 3.10], Ψ_ϕ is independent of the sequence of flips and well-defined. Finally we get an action of the mapping class group on the derived category:

$$\Psi\colon \mathrm{MCG}(\Sigma, M) \to \mathrm{Aut}(\mathcal{D}(\tau)).$$

[3]Since we have two derived equivalences, we have to choose one of them. Given a sequence of flips, we have a canonical choice. See [Nag, §2.2].

2.3 Cluster transformation

We put $T = T(\tau) := \mathbb{C}(x_i)_{i\in I}$. We regard this as the fractional field of the group ring of the Grothendieck group of the derived category $\mathcal{D}(\tau)$.

We define $\mathrm{CT}_k \colon T(f_k(\tau)) \xrightarrow{\sim} T(\tau)$ by

$$\mathrm{CT}_k(x'_i) = \begin{cases} (x_k)^{-1}\Big(\prod(x_j)^{Q(j,k)} + \prod(x_j)^{Q(k,j)}\Big) & i = k, \\ x_i & i \neq k \end{cases}$$

where $Q(i,k)$ is the nuber of arrows from i to k and x'_i is the generator of $T(f_k(\tau))$.

In the same way as the previous section, we get

$$\mathrm{CT}_\phi \colon T(\phi(\tau)) \xrightarrow{\sim} T(\tau).$$

Under the identification $T(\phi(\tau)) = T(\tau)$ induced by Ψ_ϕ, we get

$$\mathrm{CT} \colon \mathrm{MCG}(\Sigma, M) \to \mathrm{Aut}(T(\tau)).$$

3 Donaldson-Thomas theory

Let J_τ be the Jacobi algebra associated to the quiver with the potential $(Q(\tau), W(\tau))$. If $\partial\Sigma$ is non-empty, then J_τ is finite dimensional ([LF09])[4].

Let P^i_τ $(i \in I)$ be the projective J_τ-module. For $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$, we define

$$\mathrm{Hilb}^i_\tau(\mathbf{v}) := \{P^i_\tau \twoheadrightarrow V \mid \underline{\dim} V = \mathbf{v}\}.$$

This is called the *Hilbert scheme*[5].

Definition. *We define* $\mathrm{DT}_\tau \colon T \xrightarrow{\sim} T$ *by*

$$\mathrm{DT}_\tau(x_i) := (x_i)^{-1} \cdot \sum_{\mathbf{v}} \mathrm{Eu}(\mathrm{Hilb}^i_\tau(\mathbf{v})) \cdot y^{-\mathbf{v}}$$

where

$$y^{-\mathbf{v}} := \prod_i (y_i)^{-v_i}, \quad y_i := \prod_j (x_i)^{Q(i,j)}.$$

As a direct application of the main theorem in [Nag], we get the following:

Theorem 4. *For any element* $\phi \in \mathrm{MCG}(\Sigma, M)$*, we have*

$$\mathrm{DT}_\tau \circ \mathrm{CT}_\phi = \mathrm{CT}_\phi \circ \mathrm{DT}_\tau.$$

[4] If $\partial\Sigma$ is empty, then we have to take a completion $\hat{T}$ of T in Theorem 4

[5] The name comes from the Hilbert scheme in algebraic geometry which parameterizes quotient sheaves of the structure sheaf.

References

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