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LOCAL B-MODEL AND MIXED HODGE STRUCTURE

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Abstract. In this note, we explain the definition of the Yukawa coupling for the local B-model proposed in [KM].

1. Introduction

Local mirror symmetry (LMS) is a variant of mirror symmetry found by Katz–Klemm–Vafa and Chiang–Klemm–Yau–Zaslow. Each example of LMS corresponds to a two dimensional reflexive polyhedron[1]. Although LMS was derived from (the ordinary) mirror symmetry of toric Calabi–Yau hypersurfaces, its main statement can be addressed in terms of the reflexive polyhedron, without referring to the Calabi–Yau threefolds.

Take a two dimensional reflexive polyhedron Δ. To this Δ, three objects can be associated. The first is a toric surface whose fan is generated by integral points of Δ. The local A-model deals with its local Gromov–Witten invariants. The second is an affine curve C_a° (or a family of affine curves) in the two dimensional algebraic torus $\mathbb{T}^2$ whose defining equation is the sum of Laurent monomials corresponding to integral points of Δ. The local B-model is about the variation of mixed Hodge structures on the relative cohomology group $H^2(\mathbb{T}^2, C_a^\circ)$ with $\mathbb{C}$-coefficients. The third is a system of differential equations called the A-hypergeometric system introduced by Gel'fand, Kapranov and Zelevinsky [GKZ]. The statement of local mirror symmetry is that the local A and B-models are related via this system. The overview is summarized in Figure 1.

The goal of this note is to explain the definition of the Yukawa coupling for the local B-model which Satoshi Minabe and I have proposed in [KM]. The Yukawa coupling is one of the most important ingredients in mirror symmetry: it is the third derivative of the generating function of genus zero Gromov–Witten invariants (the prepotential) in the A-model, and it is the third derivative of the period map for the family of mirror Calabi–Yau threefolds in the B-model. As to our knowledge, there has been no direct definition of the Yukawa coupling for the local B-model although physicists computed it in several cases. Our definition is based on Batyrev's and Stienstra's results [B, S] on the variation of mixed Hodge structures on $H^2(\mathbb{T}^2, C_a^\circ)$ and it agrees with the previous computations by physicists.

[1]In [CKYZ], wider class of two dimensional convex polyhedra are also considered. But in this article, we restrict to the reflexive case.

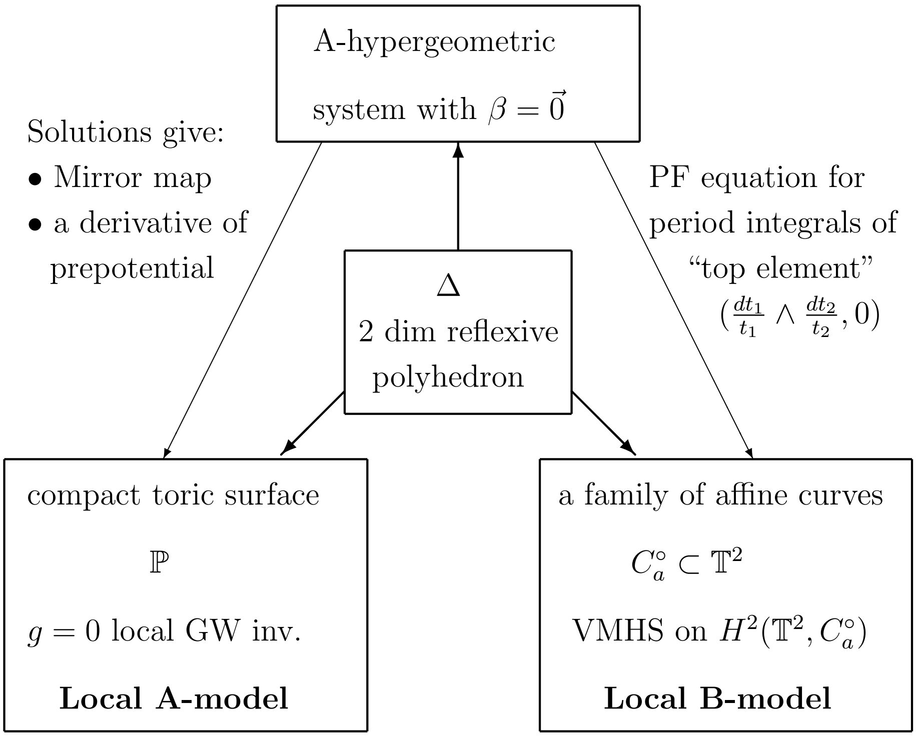

FIGURE 1. Overview of Local mirror symmetry.

Notations: For any variable x, $\theta_x := x\frac{\partial}{\partial x}$ is the logarithmic derivative by x. For $m = (m_1, m_2) \in \mathbb{Z}^2$ and variables t_1, t_2, $t_1^{m_1} t_2^{m_2}$ is denoted by the shorthand notation t^m. For a connection ∇ on a vector bundle and a vector field ∂_x, $\nabla_x := \nabla_{\partial_x}$.

2. "Jacobian ring"

For the materials in Sections §2– §4, see [B] for details.

A two dimensional polyhedron $\Delta \subset \mathbb{Z}^2 \otimes \mathbb{R}$ is a reflexive polyhedron if it is the convex hull of a finite number of integral points; the origin 0 is contained in Δ; and the distance between 0 and every edge is one, i.e. every edge is written as

$$\{(m_1, m_2) \in \Delta \mid c_1 m_1 + c_2 m_2 = 1\}$$

with some integers c_1, c_2 prime to each other. There are 16 such polyhedra.

Let $\Delta \subset \mathbb{Z}^2 \otimes \mathbb{R}$ be a two dimensional reflexive polyhedron. By associating to each integral point $m = (m_1, m_2) \in \Delta$ the Laurent monomial $t^m = t_1^{m_1} t_2^{m_2}$, we have the Laurent polynomial

$$F_a(t_1, t_2) := \sum_{m \in \Delta \cap \mathbb{Z}^2} a_m t^m \quad \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \ .$$

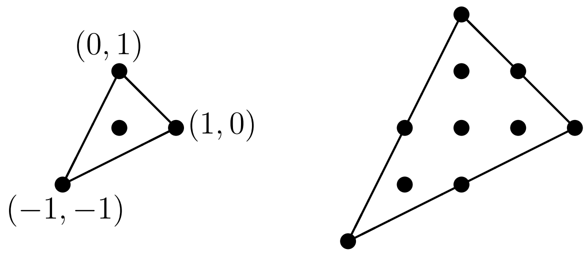

FIGURE 2. An example of reflexive polyhedron $\Delta_{\mathbb{P}^2}$ (left) and $\Delta_{\mathbb{P}^2}(2)$ (right).

Here $a = \{a_m\}$ are complex parameters. This $F(t)$ is said to be Δ-regular if for every l-dimensional face $\Delta' \subset \Delta$ ($l = 0, 1, 2$), the equations

$$F^{\Delta'} := \sum_{m \in \Delta' \cap \mathbb{Z}^2} a_m t^m = 0 \ , \quad \frac{\partial F^{\Delta'}}{\partial t_1} = 0 \ , \quad \frac{\partial F^{\Delta'}}{\partial t_2} = 0$$

have no common solutions in $\mathbb{T}^2$. The space of complex parameters a satisfying the Δ-regularity condition is denoted by $\mathbb{L}_{\rm reg}$:

$$\mathbb{L}_{\rm reg} := \{(a_m)_{m \in \Delta \cap \mathbb{Z}^2} \in \mathbb{C}^{\#(\Delta \cap \mathbb{Z}^2)} \mid \ F_a \text{ is } \Delta\text{-regular}\} \ . \tag{2.1}$$

Let $\Delta(k)$ ($k \in \mathbb{Z}_{\geq 0}$) be the polytope obtained from Δ by enlarging k-times:

$$\Delta(k) := \left\{ m \in \mathbb{R}^2 \ \middle| \ \frac{m}{k} \in \Delta \right\} \ (k \geq 1) \ , \quad \Delta(0) := \{0\} \ .$$

Let S_k ($k \in \mathbb{Z}_{\geq 0}$) be the vector space spanned by the Laurent monomials corresponding to integral points of $\Delta(k)$, i.e.

$$S_k = \bigoplus_{m \in \Delta(k) \cap \mathbb{Z}^2} \mathbb{C}\, t_0^k t^m \ .$$

Define S_Δ as the direct sum of S_k ($k \geq 0$):

$$S_\Delta = \bigoplus_{k \geq 0} S_k \quad \subset \mathbb{C}[t_0, t_1^{\pm 1}, t_2^{\pm 1}] \ .$$

Consider the derivations D_0, D_1, D_2 acting on S_Δ in the following way:

$$D_0(t_0^k t^m) := (k + t_0 F_a) t_0^k t^m \ ,$$
$$D_i(t_0^k t^m) := \big(m_i + t_0(\theta_{t_i} F_a)\big) t_0^k t^m \quad (i = 1, 2) \ .$$

The quotient vector space which we introduce in the next definition will play a key role.

Definition 2.1.

$$\mathcal{R}_{F_a} := S_\Delta / \sum_{i=0}^{2} D_i S_\Delta \ .$$

For $a \in \mathbb{L}_{\rm reg}$, $\mathcal{R}_{F_a}$ is a finite dimensional vector space and its dimension is equal to the volume of Δ.

Example 2.2. Let $\Delta_{\mathbb{P}^2}$ be the convex hull of $(1,0),(0,1),(-1,-1)$ shown in Figure 2. This is the two dimensional reflexive polyhedron with the least number of integral points. (The reason for the subscript $\mathbb{P}^2$ is because the corresponding toric surface in the local A-model is the complex projective plane $\mathbb{P}^2$.) The associated Laurent polynomial is

$$F_a = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2} \ ,$$

and the Δ-regularity condition is the following:

$$\mathbb{L}_{\rm reg} = \{(a_0,a_1,a_2,a_3) \in \mathbb{C}^4 \mid a_1 a_2 a_3 \neq 0 \ , a_0^3 + 27 a_1 a_2 a_3 \neq 0\} \ .$$

A basis of $\mathcal{R}_{F_a}$ can be found by studying the subspace $\sum_i D_i S_\Delta$ closely. We write $\alpha \equiv \beta$ if $\alpha - \beta \in \sum_i D_i S_\Delta$ for $\alpha, \beta \in S_\Delta$. First, by operating D_0, D_1, D_2 on $1 \in S_\Delta$, we have

$$\begin{aligned} 0 &\equiv D_0 1 = t_0 F_a = a_0 t_0 + a_1 t_0 t_1 + a_2 t_0 t_2 + a_3 \frac{t_0}{t_1 t_2} \ , \\ 0 &\equiv D_1 1 = t_0 \theta_{t_1} F_a = a_1 t_0 t_1 - a_3 \frac{t_0}{t_1 t_2} \ , \\ 0 &\equiv D_2 1 = t_0 \theta_{t_2} F_a = a_2 t_0 t_2 - a_3 \frac{t_0}{t_1 t_2} \ . \end{aligned}$$

These imply that any t_0-degree one monomial in S_Δ is equal to a multiple of t_0 in $\mathcal{R}_{F_a}$:

$$a_1 t_0 t_1 \equiv a_2 t_0 t_2 \equiv a_3 \frac{t_0}{t_1 t_2} \equiv -\frac{1}{3} a_0 t_0 \ .$$

Next operating D_0, D_1, D_2 on degree one monomials $t_0, t_0 t_1, t_0 t_2, t_0/t_1 t_2$, we can easily see that any degree two monomial is equivalent to a linear combination of t_0 and t_0^2 mod $\sum_i D_i S_\Delta$. Similarly, by operating D_0, D_1, D_2 on monomials with t_0 degree ≥ 2, we see that any monomial of t_0-degree ≥ 3 is also expressed in terms of t_0, t_0^2. For example,

$$\begin{aligned} &t_0^2 t_1 \equiv -\frac{1}{3a_1} t_0 - \frac{a_0}{3a_1} t_0^2 \ , \quad t_0^2 t_1^2 \equiv \frac{4a_0}{9a_1^2} t_0 + \frac{a_0^2}{9a_1^2} t_0^2 \ , \quad t_0^2 t_1 t_2 \equiv \frac{a_0}{9a_1 a_2} t_0 + \frac{a_0^2}{9a_1 a_2} t_0^2 \ , \\ &t_0^3 \equiv \frac{-1}{\delta}(a_0 t_0 + 3a_0^2 t_0^2) \ , \quad t_0^3 t_1 \equiv \frac{a_0^2}{3a_1 \delta} t_0 + \frac{a_0^3 - 54 a_1 a_2 a_3}{3 a_1 \delta} \, t_0^2, \end{aligned} \tag{2.2}$$

where $\delta := a_0^3 + 27 a_1 a_2 a_3$. Thus for $\Delta_{\mathbb{P}^2}$, we obtain

$$\mathcal{R}_{F_a} \cong \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2 \ .$$

In Section 4, we will see that $\mathcal{R}_{F_a}$ is isomorphic to the relative cohomology group $H^2(\mathbb{T}^2, C_a^\circ)$. In this sense, $\mathcal{R}_{F_a}$ is an analog of the Jacobian ring for the cohomology group of hypersurfaces in projective spaces, hence comes the title of this section.

Remark 2.3. Although S_Δ is a ring, $\mathcal{R}_{F_a}$ does not inherit the ring structure since the subspace $\sum_{i=0}^2 D_i S_\Delta$ is not an ideal. If we consider the ideal J_{F_a} of S_Δ generated by $t_0 F_a, t_0 \theta_{t_1} F_a, t_0 \theta_{t_2} F_a$ instead of $\sum_{i=0}^2 D_i S_\Delta$, we obtain the ring S_Δ / J_{F_a}. This is isomorphic to $\mathcal{R}_{F_a}$ as a vector space.

3. "Gauss–Manin connection"

So far, the complex parameter a is fixed. From now on, we move a in $\mathbb{L}_{\mathrm{reg}}$ (defined in (2.1)) and consider the family $\mathcal{R}$ of $\mathcal{R}_{F_a}$'s:

$$\mathcal{R} := \bigcup_{a \in \mathbb{L}_{\mathrm{reg}}} \mathcal{R}_{F_a} \to \mathbb{L}_{\mathrm{reg}}$$

This is a vector bundle over $\mathbb{L}_{\mathrm{reg}}$ of rank $\mathrm{vol}(\Delta)$.

We define a connection on $\mathcal{R}$ as follows. Consider the trivial (infinite dimensional) vector bundle $S_\Delta \times \mathbb{L}_{\mathrm{reg}} \to \mathbb{L}_{\mathrm{reg}}$ with the nontrivial connection $\mathcal{D}$ given by

$$\mathcal{D}_{a_m} = \partial_{a_m} + t_0 t^m \quad (m \in \Delta \cap \mathbb{Z}^2) .$$

Since $\mathcal{D}_{a_m} D_i = D_i \mathcal{D}_{a_m}$ ($m \in \Delta \cap \mathbb{Z}^2$, $i = 0, 1, 2$), $\mathcal{D}$ induces the connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ given by

$$\nabla^{\mathcal{R}}_{a_m} [t_0^k t^m] := [\mathcal{D}_{a_m} t_0^k t^m] .$$

Here $[\alpha]$ denotes the image of $\alpha \in S_\Delta$ in $\mathcal{R}_{F_a}$.

Example 3.1. For $\Delta_{\mathbb{P}^2}$,

$$\nabla^{\mathcal{R}}_{a_0}(1, t_0, t_0^2) = (t_0, t_0^2, t_0^3) \overset{\text{eq.(2.2)}}{=} (1, t_0, t_0^2) A_0 ,$$
$$\nabla^{\mathcal{R}}_{a_i}(1, t_0, t_0^2) = (t_0 t_i, t_0^2 t_i, t_0^3 t_i) \overset{\text{eq.(2.2)}}{=} (1, t_0, t_0^2) A_i \quad (i = 1, 2, 3) ,$$

where the matrices A_0, A_1, A_2, A_3 are given by

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{-a_0}{\delta} \\ 0 & 1 & \frac{-3a_0^2}{\delta} \end{pmatrix} , \qquad A_i = \begin{pmatrix} 0 & 0 & 0 \\ \frac{-a_0}{3a_i} & \frac{-1}{3a_i} & \frac{a_0^2}{a_i \delta} \\ 0 & \frac{-a_0}{3a_i} & \frac{a_0^3 - 54 a_1 a_2 a_3}{3 a_i \delta} \end{pmatrix} \quad (i = 1, 2, 3) .$$

In Section 4, we will see that $\mathcal{R}_{F_a}$ is isomorphic to the relative cohomology group $H^2(\mathbb{T}^2, C_a^\circ)$ and that the connection $\nabla^{\mathcal{R}}$ is nothing but the Gauss–Manin connection.

4. Geometric meaning of $\mathcal{R}_{F_a}$ and $\nabla^{\mathcal{R}}$

Consider the affine curve

$$C_a^\circ = \{(t_1, t_2) \in \mathbb{T}^2 \mid F_a(t_1, t_2) = 0\} .$$

The Δ-regularity condition (i.e. $a \in \mathbb{L}_{\mathrm{reg}}$) implies that C_a° and its compactification[2] C_a are both smooth. It is not difficult to see that the genus of C_a is one, and that $C_a^\circ = C_a \setminus \{\mathrm{vol}(\Delta)\text{-points}\}$.

[2] There exists a canonical compactification of C_a° in a certain toric surface determined by the polyhedron Δ.

Consider the relative cohomology group $H^2(\mathbb{T}^2, C_a^\circ)$ of the pair $(\mathbb{T}^2, C_a^\circ)$. Note that there is an exact sequence of the mixed Hodge structures

$$0 \longrightarrow H^1(C_a^\circ)/H^1(\mathbb{T}^2) \longrightarrow H^2(\mathbb{T}^2, C_a^\circ) \longrightarrow \underbrace{H^2(\mathbb{T}^2)}_{\cong \mathbb{C}\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}} \longrightarrow 0 \ .$$

Therefore $H^2(\mathbb{T}^2, C_a^\circ)$ is an extension of $H^2(\mathbb{T}^2)$ by $PH^1(C_a^\circ) := H^1(C_a^\circ)/H^1(\mathbb{T}^2)$.

4.1. **Mixed Hodge structure on $H^2(\mathbb{T}^2, C_a^\circ)$.** For $a \in \mathbb{L}_{\mathrm{reg}}$, it is known that $\mathcal{R}_{F_a}$ is isomorphic to $H^2(\mathbb{T}^2, C_a^\circ)$ [B, S]. The isomorphism is given by

$$\begin{aligned} 1 &\mapsto \Big(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0\Big) =: \omega \ , \\ t_0^k t^m &\mapsto \Big(0, \mathrm{Res}_{F_a=0} \frac{(-1)^{k-1}(k-1)!t^m}{F_a^k} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}\Big) \quad (k \geq 1, m \in \Delta(k)) \ . \end{aligned} \tag{4.1}$$

They also showed that both the Hodge and the weight filtrations on $H^2(\mathbb{T}^2, C_a^\circ)$ are given by two filtrations of $\mathcal{R}_{F_a}$.

- **Hodge filtration:** Let $\mathcal{E}^{-i}$ $(i = 0, 1, 2, 3, \ldots)$ be the subspace of S_Δ spanned by all monomials $t_0^k t^m$ with t_0-degree $\leq i$:

$$\mathcal{E}^{-i} := \bigoplus_{k=0}^{i} S_k \ .$$

  Then the filtration $0 \subset \mathcal{E}^0 \subset \mathcal{E}^{-1} \subset \mathcal{E}^{-2} \subset \cdots$ on S_Δ induces the filtration on $\mathcal{R}_{F_a}$ (which is also denoted by $\mathcal{E}^\bullet$):

$$0 \subset \mathcal{E}^0 \subset \mathcal{E}^{-1} \subset \mathcal{E}^{-2} = \mathcal{R}_{F_a} \ .$$

  The Hodge filtration $F^2 \subset F^1 \subset F^0 = H^2(\mathbb{T}^2, C_a^\circ)$ corresponds to this filtration by the isomorphism (4.1).
- **Weight filtration:** Let I_1 be the subspace of S_Δ spanned by all monomials $t_0^k t^m$ $(k \geq 1,\ m \in \Delta(k))$ such that m is an interior points of $\Delta(k)$. Let I_3 be the subspace of S_Δ spanned by all monomials with t_0-degree ≥ 1. (So $S_\Delta = \mathbb{C}\,1 \oplus I_3$.) The filtration

$$0 \subset I_1 \subset I_3 \subset S_\Delta$$

  induces a filtration on $\mathcal{R}_{F_a}$. This corresponds to the weight filtration

$$\mathcal{W}_1 \subset \mathcal{W}_2 = \mathcal{W}_3 \subset \mathcal{W}_4 = H^2(\mathbb{T}^2, C_a^\circ) \ .$$

Example 4.1. For $\Delta_{\mathbb{P}^2}$, the Hodge and weight filtrations are summarized in Figure 3.

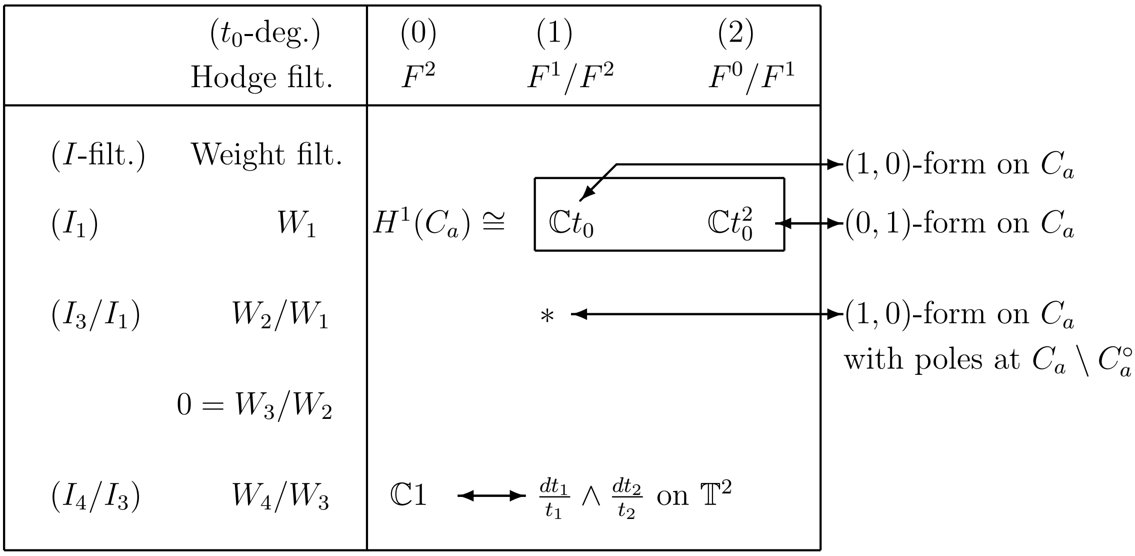

FIGURE 3. The mixed Hodge structures on $H^2(\mathbb{T}^2, C_a^\circ)$ for the case of $\Delta_{\mathbb{P}^2}$. Note that $*$ is zero dimensional for this case, but for other two dimensional reflexive polyhedra Δ, $*$ is $(\mathrm{vol}(\Delta) - 3)$-dimensional.

4.2. **Gauss–Manin connection.** Batyrev's and Stienstra's results also include results on the Gauss–Manin connection of $H^2(\mathbb{T}^2, C_a^\circ)$ for the family of affine curves

$$\mathcal{C}^\circ := \bigcup_{a \in \mathbb{L}_{\mathrm{reg}}} C_a^\circ \to \mathbb{L}_{\mathrm{reg}} \ .$$

The Gauss–Manin connection ∇ corresponds to the connection $\nabla^{\mathcal{R}}$ of $\mathcal{R}$ under the isomorphism (4.1).

Example 4.2. Consider the case of $\Delta_{\mathbb{P}^2}$. Let $\omega \in H^2(\mathbb{T}^2, C_a^\circ)$ be the element corresponding to $1 \in \mathcal{R}_{F_a}$. Recall that $\mathcal{R}_{F_a}$ is spanned by

$$1\ , \qquad t_0 = \nabla^{\mathcal{R}}_{a_0} 1\ , \qquad t_0^2 = (\nabla^{\mathcal{R}}_{a_0})^2 1\ .$$

This fact implies that $H^2(\mathbb{T}^2, C_a^\circ)$ is spanned by the corresponding elements ω, $\nabla_{a_0}\omega$ and $(\nabla_{a_0})^2\omega$. The action of the Gauss–Manin connection is the same as Example 3.1:

$$\nabla_{a_i}(\omega, \nabla_{a_0}\omega, (\nabla_{a_0})^2\omega) = (\omega, \nabla_{a_0}\omega, (\nabla_{a_0})^2\omega) A_i \quad (i = 0, 1, 2, 3)\ .$$

5. Yukawa coupling for local B-model

Now we are ready to give a definition of the Yukawa coupling.

Let $\omega \in H^2(\mathbb{T}^2, C_a^\circ)$ be the element corresponding to $1 \in \mathcal{R}_{F_a}$ by the isomorphism (4.1). Let $T^0\mathbb{L}_{\mathrm{reg}}$ be the subbundle of holomorphic tangent bundle $T\mathbb{L}_{\mathrm{reg}}$ generated by the vector field ∂_{a_0}.

Definition 5.1. The local B-model Yukawa coupling is the $\mathcal{O}_{\mathbb{L}_{\mathrm{reg}}}$-multilinear map

$$\mathrm{Yuk} : T^0\mathbb{L}_{\mathrm{reg}} \times T\mathbb{L}_{\mathrm{reg}} \times T\mathbb{L}_{\mathrm{reg}} \longrightarrow \mathcal{O}_{\mathbb{L}_{\mathrm{reg}}};$$
$$(\partial_{a_0}, \partial_{a_m}, \partial_{a_n}) \mapsto -\sqrt{-1}\int_{C_a} \nabla_{a_0}\omega \wedge \nabla_{a_m}\nabla_{a_n}\omega \ .$$

Here C_a is the compactification of the affine curve C_a°.

That the right hand side is well-defined can be shown by looking at the Hodge and the weight filtrations. First, note that $\nabla_{a_0}\omega \in F^1 \cap W_1$, or in other words, $\nabla_{a_0}\omega$ is a $(1,0)$-form on C_a. This is because it corresponds to $\nabla^{\mathcal{R}}_{a_0}1 = t_0 \in \mathcal{R}_{F_a}$, which has the t_0-degree one and lies in the interior of $\Delta(1)$. Second, note that $\nabla_{a_m}\nabla_{a_n}\omega \in F^0 \cap W_2$, or in other words, it is the sum of a $(0,1)$-form on C_a and a $(1,0)$-form on C_a which may have poles on $C_a \setminus C_a^\circ$. This is because it corresponds to $\nabla^{\mathcal{R}}_{a_m}\nabla^{\mathcal{R}}_{a_n}1 = t_0^2 t^{m+n} = f(a)t_0^2 + g(a)t_0 t^l \in \mathcal{R}_{F_a}$ (here $f(a), g(a)$ are some functions in a and $l \in \Delta(1)$). Therefore the integrand is a $(1,1)$-form on C_a which does not have poles on C_a, and hence the integration is well-defined.

Example 5.2. An easiest way to calculate the Yukawa coupling is to derive differential equations it satisfies. We demonstrate the computation of the Yukawa coupling $\mathrm{Yuk}(\theta_{a_0}, \theta_{a_0}, \theta_{a_0})$ for the case of $\Delta_{\mathbb{P}^2}$.

For the sake of simplicity, we write Y for $\mathrm{Yuk}(\theta_{a_0}, \theta_{a_0}, \theta_{a_0})$. First note that the equality $(\mathcal{D}_{a_1} - \mathcal{D}_{a_3})1 = t_0\theta_{t_1}F_a \equiv 0$ in S_Δ implies the equality $(\nabla_{a_1} - \nabla_{a_3})\omega = 0$ in $H^2(\mathbb{T}^2, C_a^\circ)$ under the isomorphism (4.1). Therefore we have the equation

$$(\theta_{a_1} - \theta_{a_3})\mathrm{Y} = \int_{C_a} \nabla_{a_0}(\nabla_{a_1} - \nabla_{a_3})\omega \wedge (\nabla_{a_0})^2\omega + \int_{C_a} \nabla_{a_0}\omega \wedge (\nabla_{a_0})^2(\nabla_{a_1} - \nabla_{a_3})\omega = 0 \ .$$

Similarly, the equations $(\theta_{a_0} + \theta_{a_1} + \theta_{a_2} + \theta_{a_3})\mathrm{Y} = (\theta_{a_2} - \theta_{a_3})\mathrm{Y} = 0$ hold. These three equations together imply that Y depends on a_0, a_1, a_2, a_3 only through the combination

$$z := \frac{a_1 a_2 a_3}{a_0^3} \ . \tag{5.1}$$

Next note the equality in S_Δ:

$$(a_0\mathcal{D}_{a_0})^3 1 = a_0^3 t_0^3 + 3a_0^2 t_0^2 + a_0 t_0 \overset{\text{eq.(2.2)}}{\equiv} \frac{3\cdot 27z}{1+27z}a_0^2 t_0^2 + \frac{27z}{1+27z}a_0 t_0 \ .$$

The corresponding equation in $H^2(\mathbb{T}^2, C_a^\circ)$ is

$$(a_0\nabla_{a_0})^3\omega = \frac{3\cdot 27z}{1+27z}(a_0\nabla_{a_0})^2\omega - \frac{2\cdot 27z}{1+27z}\cdot a_0\nabla_{a_0}\omega \ .$$

Therefore we have

$$\theta_{a_0}\mathrm{Y} = \underbrace{\int_{C_a} (a_0\nabla_{a_0})^2\omega \wedge (a_0\nabla_{a_0})^2\omega}_{=0} + \int_{C_a} a_0\nabla_{a_0}\omega \wedge (a_0\nabla_{a_0})^3\omega = \frac{3\cdot 27z}{1+27z}\mathrm{Y} \ .$$

By (5.1), we have $\theta_{a_0}\mathrm{Y} = -3\theta_z\mathrm{Y}$. Therefore the above equation becomes the ordinary differential equation $\frac{d}{dz}\mathrm{Y} = -27\mathrm{Y}/(1+27z)$. Thus we obtain

$$\mathrm{Yuk}(\theta_{a_0}, \theta_{a_0}, \theta_{a_0}) = \frac{\mathrm{const.}}{1+27z} \ .$$

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