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The Clifford index of line bundles on 2-elementary K3 surfaces

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1 Notation and conventions

• A surface means a smooth projective surface. We call a regular surface X a K3 surface if the canonical line bundle of X is trivial. We call the pair (X, L) consisting of a K3 surface X and a base point free and big line bundle L a polarized K3 surface.

• A lattice S is a free abelian group of finite rank equipped with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$. The bilinear form $\langle\ ,\ \rangle$ on S determines a canonical embedding $S \subset S^* := \mathrm{Hom}(S, \mathbb{Z})$. A lattice S is called *2-elementary* if $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^a$, where a is the minimal number of generators of S^*/S.

• For a surface X, we denote the Neron-Severi lattice of X by S_X and denote the Picard number of X by $\rho(X)$.

• A curve means a smooth projective curve. For a curve C, we define the Clifford index of a line bundle L on C by

$$\mathrm{Cliff}(L) := \deg(L) - 2(h^0(L) - 1).$$

• A line bundle L is said to *contribute to the Clifford index* of C if $h^0(L) \geq 2$ and $h^1(L) \geq 2$. We define the Clifford index of C by

$$\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(L) \mid L \text{ contributes to the Clifford index of } C\}.$$

2 Known results

Theorem 1 (Green and Lazarsfeld). Let X be a K3 surface, and let L be a base point free and big line bundle on X. Then the Clifford index of the smooth curves of $|L|$ is constant, and if $\mathrm{Cliff}(C) < [\frac{g(C)-1}{2}]$, then there exists a divisor D on X such that $\mathrm{Cliff}(D|_C) = \mathrm{Cliff}(C)$ for any curve $C \in |L|$.

By Theorem 1, the Clifford index of a base point free and big line bundle L on a K3 surface X can be defined as the Clifford index of curves of the linear system of $|L|$.

Theorem 2 (Knutsen). Let X and L be as in Theorem 1. Let c be the Clifford index of L. If $c < [\frac{g(L)-1}{2}]$, then there exists a divisor D on X which satisfies the following conditions.

(C1) $c = D(L-D) - 2$,
(C2) $L.D \leq L(L-D)$,
(C3) $h^1(D) = h^1(L-D) = 0$,
(C4) If Δ is the base divisor of $L-D$, then $L.\Delta = 0$,
(C5) $|D|$ is base point free and its general member is smooth.

In general, a divisor D satisfying the conditions from (C1) to (C3) in Theorem 2 is called *Clifford divisor*. Moreover if a Clifford divisor D satisfies the conditions (C4) and (C5), we call D a *free Clifford divisor* for L.

Theorem 3 (Schreyer). Let X and L be as in Theorem 1, and let D be a free Clifford divisor for L. Let $\{D_a\}_{a\in\mathbb{P}^1}$ be a sub-pencil of $|D|$. Then

$$\mathcal{T} = \mathcal{T}(c, D, \{D_a\}_{a\in\mathbb{P}^1}) := \bigcup_{a\in\mathbb{P}^1} \bar{D}_a$$

is a $(h^0(L) - h^0(L-D))$-dimensional rational normal scroll of degree $h^0(L-D)$ which contains the projective model of the polarized K3 surface (X, L). Here, φ_L is the natural map defined by L, and $\bar{D}_a$ is the subspace of $\mathbb{P}^{g(L)}$ spaned by $\varphi_L(D_a)$. c is the Clifford index of L.

3 Question

The scroll $\mathcal{T} = \mathcal{T}(c, D, \{D_a\}_{a\in\mathbb{P}^1})$ is smooth if and only if $D^2 = 0$ and there is no rational curve Γ such that $L.\Gamma = 0$ and $D.L > 0$.

Question. How many smooth scrolls $\mathcal{T}(c, D, \{D_a\}_{a\in\mathbb{P}^1})$ which can be constructed by the way as in Theorem 3 are there for L?

This problem is essentially how many elliptic curves there are which compute the Clifford index of L on X. Since the Clifford index of curves of Clifford-dimension one on X can be not necessarily computed by an elliptic curve on X, this problem is not trivial.

Example (M. Reid). If there is a genus 2 curve B on X such that $L \sim rB$ $(r \geq 4)$, then the Clifford-dimension of any curve of the linear system $|L|$ is one and there is no elliptic curve which computes the Clifford index of L.

4 2-elementary K3 surface

We call a K3 surface X a *2-elementary K3 surface* if the Neron-Severi lattice S_X is a 2-elementary lattice. If X is a 2-elementary K3 surface, there exists a unique non-symplectic involution θ called canonical involution which acts trivially on S_X. Moreover, it is well known that the set of fixed points of θ is as follows.

Theorem 4 (Nikulin). Let X be a 2-elementary K3 surface and θ be as above. Then the set of fixed points X^θ has the form

$$X^\theta = \begin{cases} \phi, & (\rho(X), a, \delta_{S_X}) = (10, 10, 0) \\ C_1^{(1)} + C_2^{(1)}, & (\rho(X), a, \delta_{S_X}) = (10, 8, 0) \\ C^{(g)} + \sum_{1\leq i\leq k} E_i, & \text{otherwise.} \end{cases}$$

Here a is the minimal number of generators of ${S_X}^*/S_X$ and $k = (\rho(X) - a)/2$. We denote by $C^{(g)}$ a curve of genus g, where $g = (22 - \rho(X) - a)/2$, and by E_i a smooth rational curve. We note that $C^{(g)}$ and E_i $(1 \leq i \leq k)$ do not intersect each other. If $g \geq 2$, then the involution θ on X is of *elliptic type*.

5 Main result

Theorem 5. Let X be a 2-elementary K3 surface with $2 \leq a = \rho(X) \leq 6$. Let L be a base point free and big line bundle on X such that $g(L) \geq 3$. If there is no elliptic curve on X which computes the Clifford index of L, then there exists a genus 2 curve B satisfying the condition that $L \sim rB$ $(r \geq 2)$.

In Theorem 5, if we remove the condition that $\rho(X) \leq 6$, there is a counterexample. For example, in the case where $a = \rho(X) = 7$, let C be a smooth curve such that $C \sim 2X^\theta$. Since $g(C) = 13$, by [ELMS], the Clifford dimension of $C \leq 2$. By the genus formula, C is not a plane curve and so the Clifford dimension of C is one. However, since, for any elliptic curve F on X, $C.F = 8$ and $(X^\theta)^2 = 6$, there is no elliptic curve which computes the Clifford index of $L = \mathcal{O}_X(C)$.