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# On projective space bundles with nef normalized tautological divisor

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**Abstract**

In this poster, I explain about the structure of projective bundles with nef normalized tautological divisor.

### Definition of normalized tautological divisor

$X$ : *sm. proj. var.* / $k = \bar{k}$, $\mathcal{E}$ : *vect. bdle of rank* $r$,

$\pi : \mathbb{P}_X(\mathcal{E}) \to X$ : *projectivization*, $\xi_{\mathcal{E}}$ : *tautol. div.*

($\lozenge$) $\Lambda_{\mathcal{E}} := \xi_{\mathcal{E}} - \frac{1}{r}\pi^*(\mathbf{det}(\mathcal{E}))$ : *norm. taut. div.*

$r\Lambda_{\mathcal{E}} = r\xi_{\mathcal{E}} - \pi^*(det(\mathcal{E})) = -K_\pi$ : *rel.anti-can.div.*

### Problem

Study the structure of projective space bundles with nef normalized tautological divisor.

### Ample case, Kollár-Miyaoka-Mori [KMM]

$\pi : Y \to X$ : *gen.smooth mor.* $\Rightarrow -K_\pi$ : *not ample*

In particular, $\Lambda_{\mathcal{E}}$ cannot be ample.

### Nef and big case, Theorem1

If $\Lambda_{\mathcal{E}}$ is nef $\Rightarrow$ $\Lambda_{\mathcal{E}}$ cannot be big.

### Trivial example ($\Lambda_{\mathcal{E}}$ is semiample)

If $\mathcal{E} \cong \mathcal{O}_X^r$, then $r\Lambda_{\mathcal{E}} = p^*(-K_{\mathbb{P}^{r-1}})$ is basepoint-free where $p : \mathbb{P}_X(\mathcal{E}) \cong X \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the second projection.

### Semiample case, Theorem2

1. $char(k) = 0$ and $\Lambda_{\mathcal{E}}$ is semiample
   $\Rightarrow \exists f : X' \to X$ ; finite étale morphism
   s.t. $f^*\mathcal{E}$ is trivial up to twist by some line bundle.
2. $char(k) > 0$ and $\Lambda_{\mathcal{E}}$ is semiample
   $\Rightarrow \exists f : X' \to X$ ; finite surj. mor. from normal var.
   s.t. $f^*\mathcal{E}$ is trivial up to twist by some line bundle.

### Nef case 1, Miyaoka [M], Nakayama [N]

$X$ : *d-dim. sm. proj. var.* / $\mathbb{C}$, $\mathcal{E}$ : *vect. bdle of rank* $r$,
Then the following conditions are equivalent:

1. $\Lambda_{\mathcal{E}}$ is nef;
2. $\mathcal{E}$ is $A$-semistable and
$$(c_2(\mathcal{E}) - \frac{2r}{r-1}c_1^2(\mathcal{E})).A^{d-2} = 0$$
   for an ample divisor $A$;
3. $$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E} \ : \ filt.of\ subbdles.$$
   s.t. $\mathcal{E}_i/\mathcal{E}_{i-1}$ are induced from a repre. $\pi_1(X) \to PU(r)$ and $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ are num. equiv. to $\mu(\mathcal{E}) := c_1(\mathcal{E})/rank(\mathcal{E})$ for any $i$.

**Corollary.** *$X$, $\mathcal{E}$ as above.*

1. *Assume that $X$ is simply connected and $\Lambda_{\mathcal{E}}$ is nef. Then $\mathcal{E}$ is trivial up to twist by a line bundle.*
2. *Assume that $\mathbb{P}_X(\mathcal{E})$ is log Fano and $\Lambda_{\mathcal{E}}$ is nef. Then $X$ is log Fano and $\mathbb{P}_X(\mathcal{E}) \cong X \times \mathbb{P}^{n-1}$.*

### Nef case 2, Theorem3

$X$ : *d-dim. sm. proj. var.of pos.char.*
$\mathcal{E}$ : *vect. bdle of rank* $r$,
Then the following conditions are equivalent:

1. $\Lambda_{\mathcal{E}}$ is nef;
2. $\mathcal{E}$ is strongly $A$-semistable and
$$(c_2(\mathcal{E}) - \frac{2r}{r-1}c_1^2(\mathcal{E})).A^{d-2} = 0$$
   for an ample divisor $A$;

*Remark.* This is proved by Y. Miyaoka[M] in the curve case and A. Langer[L] in the case where $det(\mathcal{E})$ is trivial.

**Corollary.** *Let $S$ be a K3 surface or an Enriques surface then $\Omega_S$ is not nef.*

### Tangent bundle, Theorem4

1. $\Lambda_{\mathcal{T}_X}$ is nef $\Rightarrow$ X contains no rational curve;
2. $S$ : sm. proj. surf. / $\mathbb{C}$. If $\Lambda_{\mathcal{T}_S}$ nef
   $\Rightarrow \exists f : A \to S$ : étale covering from abelian surface A.

*Remark.* This is proved by I. Biswas[B] independently and recently proved by P. Jahnke and I. Radloff in arbitrary dimension.

### References

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