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Mirror symmetry for complete intersection Calabi-Yau threefolds in Gorenstein minuscule Schubert varieties

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1 Minuscule Schubert varieties

Let $G$ be a simply-connected simple complex algebraic group, $B$ a Borel subgroup and $T$ a maximal torus such that $T \subseteq B \subseteq G$. We denote by $W$ the associated Weyl group. We also fix a parabolic subgroup $P \supseteq B$. Let us denote by $W_P$ the Weyl group of $P$. We denote by $W_P$ the Weyl group of $P$ and by $W$ the set of minimal length representatives in $W$ of the coset $W/P$. For any $w \in W$, we denote with $X(w) = B/(P/P)$ the Schubert variety in $G/P$ associated to $w$.

We assume that $P = P_0$ a maximal parabolic subgroup with associated fundamental weight $\omega$, and $\omega$ is minuscule.

The minuscule homogeneous spaces $G/P_0$ are the Grassmannians, the quadrics, and the orthogonal Grassmannians, and other exceptional varieties: the Cayley plane $OP^2 = E_6(3)$, the Freudenthal variety $E_7(2)$, the Lagrangian Grassmannians, and two exceptional varieties: the Cayley plane $OP^2 = E_6(3)$ and the Freudenthal variety $E_7(2)$. The Schubert varieties in $G/P_0$ are then called minuscule Schubert varieties.

For the minuscule Schubert varieties $X(w)$, the Picard group is generated by a very ample invertible sheaf $\mathcal{O}_X$, and the basis of $\mathcal{O}_X$ is parameterized by a distributive lattice $H_0 \subseteq WP$. Now we take $V = X(w)$ a particular Schubert variety in Cayley plane $OP^2$ whose associated poset $D(H_0)$ is the following.

Theorem 1.1. There exists a smooth linear section Calabi-Yau threefold $X$ in $V$. Its topological invariants are 

$$
\text{deg}(X) = 33, \quad c_2(X) \cdot H = 78, \quad \chi(X) = -102.
$$

Remark 1.2. This $X$ is the essentially unique nontrivial example of smooth complete intersection Calabi-Yau threefolds in Gorenstein minuscule Schubert varieties. Another example is just a complete intersection in orthogonal Grassmannian $OG(5, 10)$.

2 Toric degenerations

Let $H = (H, \leq)$ be a finite distributive lattice and $D = D(H)$ the finite poset of nonzero join-irreducible elements of $H$. An element of $H$ can be regarded as a lower subset in $D$.

Let us denote by $S$ the set of stars, the additional maximal and minimal elements for $D$, and by $E$ the set of oriented edges in the Hasse diagram of $D \cup S$. We define a linear map $\lambda : L(H) \rightarrow L(E)$ as

$$
\lambda(\tau)(e) := \begin{cases}
1, & \text{if } e \in \tau \\
0, & \text{otherwise}
\end{cases}
$$

where $\tau \in H$ is regarded as a lower subset in $D \cup S$ and the head $h(e)$ and the tail $t(e)$ in $D \cup S$ are defined as follows:

$$
t(e) := \sum_{\gamma \in H} h(e), \quad \partial(e) := h(e) - t(e).
$$

The polytope $\Delta_H := \text{Conv}(\lambda(H)) \subseteq L(E)^*$ is actually defined in $M_\mathbb{Z} := L(D)^* \subseteq L(E)^*$. The following lemma gives a combinatorial description of degenerations of Gonciulea-Lakshmibai.

Lemma 2.1. Let $H = (H, \leq)$ be a distributive lattice, and assume that the $k$-algebra $R = k[p_1, \ldots, p_n]$, $\alpha \in H$, $\beta \in H$ satisfies the standard monomial basis $\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$ and the ideal $I$ is generated by degree two starthing relations:

$$
\rho \tau \phi \text{ for non-comparable in } H. \text{ Then } \text{Proj} R \text{ degenerates to a normal toric variety } P_{\Delta_H}.
$$

Furthermore, if the poset $D \cup S$ has the constant maximal length $c_1$ of oriented paths, we can see the duality of reflexive polytopes:

$$
\delta : L(E) \rightarrow L(D)^* \subseteq L(S)^* \oplus L(D)^*,
$$

where $\delta$ is the composition of boundary map $\partial$ and projection $pr_1$.

Theorem 2.2. A Gorenstein minuscule Schubert variety $X(w)$ degenerates to the Gorenstein toric Fano variety $P_{\Delta_H}$.

Remark 2.3. It is also true for the Lagrangian Grassmannians with other distributive lattices.

3 Mirror symmetry for $X$

The mirror construction is based on the idea of conifold transitions. We present the power series expansion of the period map in terms of the dual graph of the Hasse diagram for $D_{\mathbb{Z}} \cup S$.

<table>
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<th>$g = 1$ for $X$</th>
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Remark 3.3. We observe the switching of BPS numbers.